THE

MESSENGER OF MATHEMATICS.

EDITED BY

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VOL. XLV.

[May 1915—April 1916].

Cambridge: BOWES & BOWES.
London: MACMILLAN & CO. Ltd.
Glasgow: JAMES MACLEHOSE & SONS.

1916.

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ERRATUM—p. 46, last line, *omit* "$=1$."

MESSENGER OF MATHEMATICS.

ON CERTAIN SOLUTIONS OF MAXWELL'S EQUATIONS.

By *H. Bateman.*

Vector-fields with moving singular curves.

§ 1. I HAVE shown elsewhere* that it is possible to obtain a solution of Maxwell's equations which represents a vector field in which the electric and magnetic intensities are infinite at a moving point Q, whose coordinates at time α are ξ, η, ζ, and also along a moving curve attached to this point; the curve being the locus of a series of points projected from the different positions of Q, and travelling along straight lines with the velocity of light. The direction of projection for any position of Q was chosen so that it made an angle θ with the tangent to the path of Q such that $c\cos\theta = v$, where v is the velocity of Q, and c the velocity of light. This condition is, however, not invariant under the transformations of the theory of relativity, and I now find that it is not necessary to restrict the direction of projection in the way described; the introduction of the restriction was due to the mistaken idea that the second of equations (291)† is a consequence of the first.

Let α and β be defined as before by the equations

$$(x-\xi)^2 + (y-\eta)^2 + (z-\zeta)^2 = c^2(t-\tau)^2 \quad \ldots\ldots(1),$$

$$l(x-\xi) + m(y-\eta) + n(z-\zeta) = c^2 p(t-\tau) \ldots\ldots(2),$$

* *The Mathematical Analysis of Electrical and Optical Wave Motion on the basis of Maxwell's Equations*, p. 128. This will be cited later as E.

† E, p. 129. The error in the proof occurs when the axis of y is chosen so that $n_1 = 0$; this introduces a restriction, since $l_1 . m_1 . n_1$ are generally complex. The same error occurs in one of my previous papers. *Annals of Mathematics* (1914).

where ξ, η, ζ, τ are functions of α only, and l, m, n, p are functions of α and β which depend linearly on β, so that

$$l=\beta l_1-l_0,\quad m=\beta m_1-m_0,\quad n=\beta n_1-n_0,\quad p=\beta p_1-p_0\ldots(3),$$

To make the values of α and β unique, we write $\tau=\alpha$ and introduce the inequality $\tau\leq t$. The quantities l, m, n, p must, moreover, be chosen so that $l^2+m^2+n^2=c^2p^2$, and so we have the relations

$$l_1^2+m_1^2+n_1^2=c^2p_1^2,\quad l_0l_1+m_0m_1+n_0n_1=c^2p_0p_1,\quad l_0^2+m_0^2+n_0^2=c^2p_0^2\ldots(4).$$

We now use the symbol f to denote an arbitrary function of α and β, and write

$$P=\xi'(x-\xi)+\eta'(y-\eta)+\zeta'(z-\zeta)-c^2(t-\alpha)\ldots.(5).$$

The vector field which will be the subject of discussion is that defined by the electro-magnetic potentials

$$A_x=\frac{lf}{P},\quad A_y=\frac{mf}{P},\quad A_z=\frac{nf}{P},\quad \Phi=\frac{cpf}{P}\ldots\ldots..(6).$$

These have been shown to be wave-functions which satisfy the relation

$$\operatorname{div}A+\frac{1}{c}\frac{\partial\Phi}{\partial t}=0.$$

On calculating the components of the electric and magnetic intensities with the aid of the relations

$$H=\operatorname{rot}A,\quad E=-\frac{1}{c}\frac{\partial A}{\partial t}-\frac{\partial\Phi}{\partial x},$$

we find, as before, that the component of the electric intensity along the radius from ξ, η, ζ, α to x, y, z, t is

$$-\frac{f}{P^2}[c^2p-l\xi'-m\eta'-n\zeta'].$$

To make the electric charge associated with the singularity ξ, η, ζ, α a constant quantity 4π, we choose f so that

$$f(c^2p-l\xi'-m\eta'-n\zeta')=\xi'^2+\eta'^2+\zeta'^2-c^2.$$

It will be convenient, however, to have a value of f independent of β, and so we shall introduce the condition

$$c^2p_1-l_1\xi'-m_1\eta'-n_1\zeta'=0;$$

as before, the value of f is then given by the equation*

$$f(c^2 p_0 - l_0 \xi' - m_0 \eta' - n_0 \zeta') = c^2 - \xi'^2 - \eta'^2 - \zeta'^2 \quad \ldots.(7).$$

We shall suppose that l_0, m_0, n_0, p_0 are real, then f is real and l_1, m_1, n_1, p_1 are generally complex quantities. Let $\bar{l}_1, \bar{m}_1, \bar{n}_1, \bar{p}_1$ be the conjugate complex quantities, then we have the relations

$$\left.\begin{array}{c} \bar{l}_1^2 + \bar{m}_1^2 + \bar{n}_1^2 = c^2 \bar{p}_1^2, \quad l_0 \bar{l}_1 + m_0 \bar{m}_1 + n_0 \bar{n}_1 = c^2 p_0 \bar{p}_1 \\ c^2 \bar{p}_1 - \bar{l}_1 \xi' - \bar{m}_1 \eta' - \bar{n}_1 \zeta' = 0 \end{array}\right\} \ldots(8).$$

Now let a set of real quantities $\bar{l}_0, \bar{m}_0, \bar{n}_0, \bar{p}_0$ be chosen so that

$$\left.\begin{array}{c} \bar{l}_0 l_1 + \bar{m}_0 m_1 + \bar{n}_0 n_1 = c^2 \bar{p}_0 p_1, \quad \bar{l}_0 \bar{l}_1 + \bar{m}_0 \bar{m}_1 + \bar{n}_0 \bar{n}_1 = c^2 \bar{p}_0 \bar{p}_1 \\ \bar{l}_0^2 + \bar{m}_0^2 + \bar{n}_0^2 = c^2 \bar{p}_0^2, \quad l_0 \bar{l}_0 + m_0 \bar{m}_0 + n_0 \bar{n}_0 - c^2 p_0 \bar{p}_0 = h \neq 0 \end{array}\right\} \ldots(9).$$

Then if we write $x - \xi = X$, $y - \eta = Y$, $z - \zeta = Z$, $t - \alpha = T$,

$$\left.\begin{array}{ll} c^2 p_1 T - l_1 X - m_1 Y - n_1 Z = S, & c^2 \bar{p}_1 T - \bar{l}_1 X - \bar{m}_1 Y - \bar{n}_1 Z = \bar{S} \\ c^2 p_0 T - l_0 X - m_0 Y - n_0 Z = U, & c^2 \bar{p}_0 T - \bar{l}_0 X - \bar{m}_0 Y - \bar{n}_0 Z = \bar{U} \end{array}\right\} \ldots\ldots(10),$$

we find that, if h is suitably chosen, there is an identity of the type

$$S\bar{S} - U\bar{U} \equiv k(c^2 T^2 - X^2 - Y^2 - Z^2) = 0 \ldots\ldots.(11),$$

where k is a function of α whose value may be determined by replacing T by 1, X by ξ', Y by η', and Z by ζ' in the identity. We thus find that

$$fk = \bar{l}_0 \xi' + \bar{m}_0 \eta' + \bar{n}_0 \zeta' - c^2 \bar{p}_0 \ldots\ldots\ldots\ldots(12).$$

By considering the relations satisfied by l_1, m_1, n_1, p_1, we see that

$$\xi' = \lambda l_0 + \mu \bar{l}_0, \quad \eta' = \lambda m_0 + \mu \bar{m}_0, \quad \zeta' = \lambda n_0 + \mu \bar{n}_0, \quad 1 = \lambda p_0 + \mu \bar{p}_0 \ldots(13),$$

where λ and μ are quantities to be determined. We deduce at once from these equations that

$$l_0 \xi' + m_0 \eta' + n_0 \zeta' - c^2 p_0 = \mu h, \quad \bar{l}_0 \xi' + \bar{m}_0 \eta' + \bar{n}_0 \zeta' - c^2 \bar{p}_0 = \lambda h,$$

$$\xi'^2 + \eta'^2 + \zeta'^2 - c^2 = 2\lambda\mu h.$$

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* It should be noticed that when this condition is satisfied the field specified by potentials of type $A_x^0 = \dfrac{lf}{P} + \dfrac{\xi'}{P}$ is conjugate to the field specified by Liénard's potentials of type $A_x' = \dfrac{\xi'}{P}$ and the relation $A_x^0 A_x' + A_y^0 A_y' + A_z^0 A_z' - \Phi^0 \Phi' = 0$ is satisfied. Compare this with the remark E, p. 135.

Hence $$2\lambda = \frac{\xi'^2+\eta'^2+\zeta'^2-c^2}{l_0\xi'+m_0\eta'+n_0\zeta'-c^2p_0} = f \quad\ldots\ldots\ldots\ldots(14).$$

The expression for A_x can now be thrown into a more convenient form. Differentiating equation (2) we obtain

$$l = \left(\beta\frac{\partial S}{\partial\alpha} - \frac{\partial U}{\partial\alpha}\right)\frac{\partial\alpha}{\partial x} + S\frac{\partial\beta}{\partial x};$$

also $\beta S = U$, hence we may write

$$A_x = \frac{f}{P}\frac{\partial\alpha}{\partial x}\left[\frac{U}{S}\frac{\partial S}{\partial\alpha} - \frac{\partial U}{\partial\alpha}\right] + \frac{Uf}{P}\frac{\partial}{\partial x}\log\left(\frac{U}{S}\right) \quad\ldots\ldots(15).$$

Now let $\overline{A}_x$ be the complex quantity conjugate to A_x, then

$$\overline{A}_x = \frac{f}{P}\frac{\partial\alpha}{\partial x}\left[\frac{\overline{U}}{\overline{S}}\frac{\partial\overline{S}}{\partial\alpha} - \frac{\partial\overline{U}}{\partial\alpha}\right] + \frac{\overline{U}f}{P}\frac{\partial}{\partial x}\log\left(\frac{\overline{U}}{\overline{S}}\right).$$

and so if $2a_x = A_x + \overline{A}_x$, we have

$$2a_x = \frac{f}{P}\frac{\partial\alpha}{\partial x}\left[U\frac{\partial}{\partial\alpha}\log(S\overline{S}) - 2\frac{\partial U}{\partial\alpha}\right] + \frac{Uf}{P}\frac{\partial}{\partial x}\log\left(\frac{U^2}{S\overline{S}}\right)\ldots(16).$$

Now differentiate the identity (11) with regard to α, keeping x, y, z, t constant, we obtain

$$\frac{\partial}{\partial\alpha}(S\overline{S}) - \frac{\partial}{\partial\alpha}(U\overline{U}) = 2kP.$$

Substituting in (16), making use of the relation $U\overline{U} = S\overline{S}$, we find that

$$2a_x = \frac{f}{P}\frac{\partial\alpha}{\partial x}\left[U\frac{\partial}{\partial\alpha}\log\left(\frac{\overline{U}}{U}\right) + 2\frac{kP}{\overline{U}}\right] - \frac{Uf}{P}\frac{\partial}{\partial x}\log\left(\frac{\overline{U}}{U}\right)$$

$$= \frac{f}{P}\frac{\partial\alpha}{\partial x}\left[U\frac{\partial}{\partial\alpha}\log\left(\frac{\mu\overline{U}}{\lambda U}\right) + \frac{2kP}{\overline{U}}\right] - \frac{Uf}{P}\frac{\partial}{\partial x}\log\left(\frac{\mu\overline{U}}{\lambda U}\right)$$

$$= \frac{fU}{\mu P\overline{U}}\frac{\partial\alpha}{\partial x}\frac{\partial}{\partial\alpha}(\mu\overline{U}) - \frac{f}{\lambda P}\frac{\partial\alpha}{\partial x}\frac{\partial}{\partial\alpha}(\lambda U) + \frac{2kf}{\overline{U}} - \frac{Uf}{P}\frac{\partial}{\partial x}\log\left(\frac{\mu\overline{U}}{\lambda U}\right).$$

Now it follows at once from (13) that $\lambda U + \mu \bar{U} \equiv -P$, and $2\lambda = f$, hence we have

$$2a_x = \frac{f\dot{U}}{\mu P\bar{U}}\frac{\partial\alpha}{\partial x}\frac{\partial}{\partial\alpha}(\mu\bar{U}) + \frac{2}{P}\frac{\partial\alpha}{\partial x}\frac{\partial}{\partial\alpha}(P+\mu\bar{U}) + \frac{2kf}{\bar{U}}\frac{\partial\alpha}{\partial x} - \frac{Uf}{P}\frac{\partial}{\partial x}\log\left(\frac{\mu\bar{U}}{\lambda U}\right)$$

$$= 2\frac{\partial\alpha}{\partial x}\frac{\partial}{\partial\alpha}\log\left(\frac{P}{\mu\bar{U}}\right) + \frac{2kf}{\bar{U}}\frac{\partial\alpha}{\partial x} - \frac{Uf}{P}\frac{\partial}{\partial x}\log\left(\frac{\mu\bar{U}}{\lambda U}\right).$$

Now $$\frac{\partial}{\partial x}(\log P) = \frac{\xi'}{P} + \frac{\partial\alpha}{\partial x}\frac{\partial}{\partial\alpha}(\log P),$$

consequently our expression for A_x can be written in the form

$$a_x = -\frac{\xi'}{P} + \frac{\partial\alpha}{\partial x}\left[\frac{kf}{\bar{U}} - \frac{\partial}{\partial\alpha}\log(\mu\bar{U})\right] + \frac{\partial}{\partial x}(\log P) - \frac{\lambda U}{P}\frac{\partial}{\partial x}\log\left(\frac{\mu\bar{U}}{\lambda U}\right).$$

Making use of the value of fk given by (12) and using $d/d\alpha$ to denote a differentiation with regard to α *when X, Y, Z, and T are regarded as constant*, we obtain the simple formula

$$a_x = -\frac{\xi'}{P} - \frac{\partial\alpha}{\partial x}\frac{d}{d\alpha}\log\bar{U} + \frac{\partial F}{\partial x} \quad \text{..........(17)},$$

where F is a function whose exact form is not needed. The corresponding expressions for a_y, a_z, and ϕ may be written down by analogy. Separating the expressions for a_x, a_y, a_z, ϕ into two parts, we obtain terms of the type $-\xi'/P$ representing the potentials of an electro-magnetic field with an electric charge 4π associated with the point singularity $(\xi, \eta, \zeta, \alpha)$, and also terms of type

$$A_x^0 = -\frac{\partial\alpha}{\partial x}\frac{d}{d\alpha}\log\bar{U} + \frac{\partial F}{\partial x} \quad \text{.............(18)}.$$

These will be regarded as the potentials of an *elementary æthereal field* whose singular curve is obtained by putting $\bar{U} = 0$. We easily find that $\bar{U} = 0$ when and only when

$$\frac{X}{l_0} = \frac{Y}{m_0} = \frac{Z}{n_0} = \frac{T}{p_0} \quad \text{..................(19)}.$$

The singular curve is thus built up of points which travel along straight lines with the velocity of light; moreover, there is no restriction on the direction of projection, for we can make $\bar{l}_0$, $\bar{m}_0$, and $\bar{n}_0$ arbitrary functions of α and construct the corresponding expressions of type (18) without having to determine l_0, m_0, n_0, p_0, for $\lambda U = -\bar{P} - \mu \bar{U}$, and μ is determined by the equation

$$2\mu(\bar{l}_0\xi' + \bar{m}_0\eta' + \bar{n}_0\zeta' - c^2\bar{p}_0) = \xi'^2 + \eta'^2 + \zeta'^2 - c^2.$$

It should be noticed that if the direction of projection does not vary with α, $\frac{1}{\bar{U}}\frac{d\bar{U}}{d\alpha}$ is either zero, or a function of α, and then the components of the electric and magnetic intensities in the æthereal field defined by (18) are null. *Hence the æthereal field exists only in regions of space and time corresponding to values of α for which the direction of projection of the singularities varies with the time.** If the direction of projection is originally constant, then varies for a short interval, and finally becomes constant again, the singular curve of the æthereal field will at any instant be of finite length.

With regard to the directions of projection specified by l_0, m_0, n_0, p_0 and $\bar{l}_0, \bar{m}_0, \bar{n}_0, \bar{p}_0$, it follows at once from (13) that they lie in a plane containing the tangent to the path of Q. If, moreover, we draw a sphere of radius c having the point Q as centre, and measure a length v in the direction of the motion of Q so as to obtain a point V for which $QV = v$, the points on the sphere which correspond to the directions l_0, m_0, n_0, $\bar{l}_0, \bar{m}_0, \bar{n}_0$, will lie on a line through V. This is an immediate consequence of equations (4), (8), and (9), which indicate that the tangent planes at the points $\left(\frac{l_0}{p_0}, \frac{m_0}{p_0}, \frac{n_0}{p_0}\right)$, $\left(\frac{\bar{l}_0}{\bar{p}_0}, \frac{\bar{m}_0}{\bar{p}_0}, \frac{\bar{n}_0}{\bar{p}_0}\right)$ of the sphere intersect in a line which meets the sphere in two points $\left(\frac{l_1}{p_1}, \frac{m_1}{p_1}, \frac{n_1}{p_1}\right)$, $\left(\frac{\bar{l}_1}{\bar{p}_1}, \frac{\bar{m}_1}{\bar{p}_1}, \frac{\bar{n}_1}{\bar{p}_1}\right)$, which lie on the polar plane of V on account of the relations of type $l_1\xi' + m_1\eta' + n_1\zeta' - c^2 p_1 = 0$.

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* Cf. E, p. 130, where equations (292) should read

$$A_x^0 = R\frac{l+\xi'}{P}, \quad A_y^0 = R\frac{m+\eta'}{P}, \quad A_z^0 = R\frac{n+\zeta'}{P}, \quad \Phi^0 = R\frac{c(p+1)}{P}.$$

The signs need correction in the succeeding equations, also in the equations on pp. 117 and 118.

Expressions for the electric and magnetic intensities.

§ 2. Since

$$\bar{l}_0 \frac{dl_0}{d\alpha} + \bar{m}_0 \frac{d\bar{m}_0}{d\alpha} + \bar{n}_0 \frac{d\bar{n}_0}{d\alpha} - c^2 \bar{p}_0 \frac{d\bar{p}_0}{d\alpha} = 0,$$

we may evidently write

$$\frac{dl_0}{d\alpha} = \epsilon l_1 + \bar{\epsilon} \bar{l}_1 + \eta \bar{l}_0, \quad \frac{d\bar{m}_0}{d\alpha} = \epsilon m_1 + \bar{\epsilon} \bar{m}_1 + \eta \bar{m}_0,$$

and two similar equations, hence

$$\frac{d}{d\alpha} \log(\bar{U}) = \eta + R \frac{\epsilon S}{\bar{U}} = \eta + R \frac{\epsilon U}{\bar{S}},$$

where ϵ and η are functions of α and the symbol R is used to denote the real part of the quantity which follows.

Now, if $\bar{\beta}$ is the complex quantity conjugate to β, we have $\bar{\beta}\bar{S} = U$, and we find that the components of the magnetic force H are

$$H_x = R\epsilon \frac{\partial(\alpha, \bar{\beta})}{\partial(y, z)}, \quad H_y = R\epsilon \frac{\partial(\alpha, \bar{\beta})}{\partial(z, x)}, \quad H_z = R\epsilon \frac{\partial(\alpha, \bar{\beta})}{\partial(x, y)}.$$

On account of the characteristic properties* of the functions α and β, we may now write, for the components of the complex vector $M = H \pm iE$,

$$M_x = \epsilon \frac{\partial(\alpha, \bar{\beta})}{\partial(y, z)} = \pm \frac{i\epsilon}{c} \frac{\partial(\alpha, \bar{\beta})}{\partial(x, t)} \quad \text{.........(20)},$$

and two similar equations. The field specified by the potentials (18) thus possesses all the characteristics of the æthereal fields described in my paper in the structure of the æther;† in particular, it is conjugate to the electro-magnetic field of Liénard's type, with ξ, η, ζ, α as a moving point charge, and Poynting's vector at x, y, z, t is along the line drawn to this point from the associated position ξ, η, ζ, α of the point charge.

The direction of Poynting's vector in a general type of field.

§ 3. Let us now determine the direction of Poynting's vector in a field specified by a complex vector M given by equations of type (20), in which $\bar{\beta}$ is replaced by β, and ϵ is

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* E, § 5, 43.

† *Bull Amer. Math. Soc.*, March, 1915.

a function of the two quantities α and β, which are defined by equations (1) and (2) with the modification that ξ, η, ζ, τ, l, m, n, p are now supposed to be functions of both α and β. As we have shown elsewhere* these equations may be replaced by two equations of type

$$z - ct = \phi + \theta(x + iy), \quad z + ct = \psi - \frac{1}{\theta}(x - iy) \ldots (21),$$

where θ, ϕ, ψ are functions of α and β provided the + sign is taken in equations of type (20).

Differentiating equations (21) with regard to x, y, z, t in turn, and writing $\theta = \theta_1 + i\theta_2$, $\lambda^{-1} = \theta_1^2 + \theta_2^2$, where θ_1 and θ_2 are real, we find that

$$\left.\begin{aligned}
&P\frac{\partial\alpha}{\partial x} + Q\frac{\partial\beta}{\partial x} + \theta_1 + i\theta_2 = 0, && R\frac{\partial\alpha}{\partial x} + S\frac{\partial\beta}{\partial x} - \lambda(\theta_1 - i\theta_2) = 0\\
&P\frac{\partial\alpha}{\partial y} + Q\frac{\partial\beta}{\partial y} + i\theta_1 - \theta_2 = 0, && R\frac{\partial\alpha}{\partial y} + S\frac{\partial\beta}{\partial y} + i\lambda(\theta_1 - i\theta_2) = 0\\
&P\frac{\partial\alpha}{\partial z} + Q\frac{\partial\beta}{\partial z} - 1 = 0, && R\frac{\partial\alpha}{\partial z} + S\frac{\partial\beta}{\partial z} - 1 = 0\\
&P\frac{\partial\alpha}{\partial t} + Q\frac{\partial\beta}{\partial t} + c = 0, && R\frac{\partial\alpha}{\partial t} + S\frac{\partial\beta}{\partial t} - c = 0
\end{aligned}\right\} (22),$$

where the values of P, Q, R, S need not be written down. These equations give

$$\left.\begin{aligned}
&\lambda\left(P\frac{\partial\alpha}{\partial x} + Q\frac{\partial\beta}{\partial x}\right) - \left(R\frac{\partial\alpha}{\partial x} + S\frac{\partial\beta}{\partial x}\right) = -2\lambda\theta_1\\
&\lambda\left(P\frac{\partial\alpha}{\partial y} + Q\frac{\partial\beta}{\partial y}\right) - \left(R\frac{\partial\alpha}{\partial y} + S\frac{\partial\beta}{\partial y}\right) = +2\lambda\theta_2\\
&\lambda\left(P\frac{\partial\alpha}{\partial z} + Q\frac{\partial\beta}{\partial z}\right) - \left(R\frac{\partial\alpha}{\partial z} + S\frac{\partial\beta}{\partial z}\right) = \lambda - 1\\
&\lambda\left(P\frac{\partial\alpha}{\partial t} + Q\frac{\partial\beta}{\partial t}\right) - \left(R\frac{\partial\alpha}{\partial t} + S\frac{\partial\beta}{\partial t}\right) = -c(\lambda + 1)
\end{aligned}\right\} \ldots (23).$$

Combining these with the equations of type

$$M_x = \epsilon\frac{\partial(\alpha, \beta)}{\partial(y, z)} = \frac{i\epsilon}{c}\frac{\partial(\alpha, \beta)}{\partial(x, t)},$$

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* *Amer. Jour. of Math.*, April, 1915.

we find at once that

$$\left.\begin{aligned} -2\lambda\theta_1 M_x + 2\lambda\theta_2 M_y + (\lambda-1) M_z = 0 \\ 2\lambda\theta_2 M_z - (\lambda-1) M_y - i(\lambda+1) M_x = 0 \end{aligned}\right\} \ldots\ldots(24).$$

Since the coefficients of the components of M in these equations are real, it follows that Poynting's vector is in the direction of the line whose direction cosines are

$$l = -\frac{2\lambda\theta_1}{1+\lambda}, \quad m = \frac{2\lambda\theta_2}{1+\lambda}, \quad n = \frac{\lambda-1}{1+\lambda} \ldots\ldots(25).$$

Now I have shown in a previous paper* that when a possible pair of complex values of α and β have been chosen, there are ∞^1 corresponding sets of real values of x, y, z, t, and these are associated with the different positions of a point which travels with the velocity of light along a straight line whose direction cosines are l, m, n. Hence, if we regard Poynting's vector as an indicator of the direction in which energy flows through the field, we may conclude that the energy in the field under consideration flows along a series of lines whose directions are given by equations (25). The direction of the flow of energy at (x, y, z, t) is, moreover, the same whatever be the form of the function ϵ.

Faraday tubes.

§4. I have shown elsewhere† that the lines of electric force in the field due to a moving point charge $(\xi, \eta, \zeta, \alpha)$ can be obtained as loci of points travelling along straight lines with the velocity of light by considering directions of projection which satisfy differential equations of type

$$\mu \frac{dl}{d\alpha} = \lambda\xi'' + (\xi' - cl)(l\xi'' + m\eta'' + n\zeta'') \ldots\ldots(26),$$

where $\lambda = c - l\xi' - m\eta' - n\zeta'$, $\mu = c^2 - \xi'^2 - \eta'^2 - \zeta'^2$, and l, m, n are the direction cosines of the direction of projection at time α. It is clear from these equations that, if $\xi'' = \eta'' = \zeta'' = 0$, l, m, and n do not vary with α. Hence, if we consider an elementary æthereal field whose singular curve is always along a line of electric force, it appears that the æthereal field only exists in those domains of x, y, z, t which correspond to values of α, for which the velocity of the point ξ, η, ζ, α is not uniform. *Hence*

* See last reference.
† *Bull. Amer. Math. Soc.*, March, 1915.

the radiation in this type of æthereal field is due to the acceleration of the point ξ, η, ζ, α.

We shall now show that the differential equations (26) are covariant under a Lorentz transformation. The idea that the Faraday lines of force or Faraday tubes are the 'fibres' of an element of the æther* is thus compatible with the theory of relativity.

The simplest way of obtaining the required result is to remark that the two-way generated by a moving line of electric force satisfies the differential equation†

$$E_x d(y, z) + E_y d(z, x) + E_z d(x, y)$$
$$- cH_x d(x, t) - cH_y d(y, t) - cH_z d(z, t) = 0,$$

and is built up of the paths of particles which are projected from different positions of the point charge and travel along straight lines with the velocity of light. The differential equation and the property just mentioned are known to be covariant under a transformation which leaves Maxwell's equations unaltered in form, and so the result follows. To obtain a direct proof, we write

$$s(1+n) = l + im, \quad \sigma(1+n) = l - im,$$

the differential equation satisfied by s is then

$$2\mu \frac{ds}{d\alpha} = (\xi'' + i\eta'')(c - \zeta') + \zeta''(\xi' + i\eta')$$

$$- 2s[c\zeta'' + i(\xi'\eta'' - \xi''\eta')] + s^2[\zeta''(\xi' - i\eta') - (\zeta' + c)(\xi'' - i\eta'')].$$

Now apply the Lorentz transformation

$$\xi = \xi_0, \quad \eta = \eta_0, \quad \zeta = \zeta_0 \cosh u - c\alpha_0 \sinh u, \quad c\alpha = c\alpha_0 \cosh u - \zeta_0 \sinh u,$$

we find that

$$\xi' = \nu \xi_0', \quad \eta' = \nu \eta_0', \quad \zeta' = \nu(\zeta_0' \cosh u - c \sinh u),$$
$$\nu(c \cosh u - \zeta_0' \sinh u) = c,$$
$$l = \rho l_0, \quad m = \rho m_0, \quad n = \rho(n_0 \cosh u - \sinh u),$$
$$\rho(\cosh u - n_0 \sinh u) = 1.$$

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* Cf. J. J. Thomson, *Recent Researches*, chap. i.

† For the theory of differential equations of this type see C. Méray, *Ann. de l'école normale*, t. xvi. (1899), p. 509; E. Cartan, *ibid.*, t. xviii. (1901), p. 250; and A. C. Dixon, *Phil. Trans*, A, vol. cxcv. (1899), p. 151. If the equation can be written in the form $d(v, w) = 0$ the equations $v = \text{const.}$, $w = \text{const.}$ represent a moving line of electric force.

Hence, if $s_0(1+n_0)=l_0+im_0$, we have $s=s_0e^u$, also

$$\mu=\nu^2\mu_0 \text{ and } \lambda=\rho\nu\lambda_0.$$

Again, since $\nu=\dfrac{d\alpha_0}{d\alpha}$, we have

$$\xi''=\nu^2\xi_0''+\nu\nu'\xi_0', \quad \eta''=\nu^2\eta_0''+\nu\nu'\eta_0', \quad \zeta''=\nu^3\zeta_0'',$$

$$\zeta_0'+c=\nu e^{-u}(\zeta_0'+c), \quad c-\zeta'=\nu e^u(c-\zeta_0'),$$

$$c\nu'=\nu^2\zeta_0''\sinh u, \quad \nu[c+(c-\zeta_0')e^u\sinh u]=ce^u,$$

$$\nu[c-(\zeta_0'+c)e^{-u}\sinh u]=ce^{-u},$$

hence it is easily seen that the differential equation for s_0 is

$$2\mu_0\frac{ds_0}{d\alpha_0}=(\xi_0''+i\eta_0'')(c-\zeta_0')+\zeta_0''(\xi_0'+i\eta_0')$$

$$-2s_0[c\zeta_0''+i(\xi_0'\eta_0''-\xi_0''\eta_0')]+s_0^2[\zeta_0''(\xi_0'-i\eta_0')-(\zeta_0'+c)(\xi_0''-i\eta_0'')],$$

which is of the same type as that satisfied by s. The differential equation for σ can be transformed in a similar way.

ON CERTAIN INFINITE SERIES.

By *S. Ramanujan.*

1. THIS paper is merely a continuation of the paper on "Some definite integrals" published in this Journal.* It deals with some series which resemble those definite integrals not merely in form but in many other respects. In each case there is a functional relation. In the case of the integrals there are special values of a parameter for which the integrals may be evaluated in finite terms. In the case of the series the corresponding results involve elliptic functions.

* pp. 10–18 of vol. xliv.

2. It can be shown, by the theory of residues, that if α and β are real and $\alpha\beta=\frac{1}{4}\pi^2$, then

$$(1)\quad \frac{\alpha}{(\alpha+t)\cosh\alpha}-\frac{3\alpha}{(9\alpha+t)\cosh 3\alpha}+\frac{5\alpha}{(25\alpha+t)\cosh 5\alpha}-\dots$$
$$+\frac{\beta}{(\beta-t)\cosh\beta}-\frac{3\beta}{(9\beta-t)\cosh 3\beta}+\frac{5\beta}{(25\beta-t)\cosh 5\beta}-\dots$$
$$=\frac{\pi}{4\cos\sqrt{(\alpha t)}\cosh\sqrt{(\beta t)}}.$$

Now let

$$(2)\quad F(n)=\left\{\frac{\alpha e^{in\alpha}}{\cosh\alpha}-\frac{3\alpha e^{9in\alpha}}{\cosh 3\alpha}+\frac{5\alpha e^{25in\alpha}}{\cosh 5\alpha}-\dots\right\}$$
$$-\left\{\frac{\beta e^{-in\beta}}{\cosh\beta}-\frac{3\beta e^{-9in\beta}}{\cosh 3\beta}+\frac{5\beta e^{-25in\beta}}{\cosh 5\beta}-\dots\right\}.$$

Then we see that, if t is positive,

$$(3)\quad \int_0^\infty e^{-2tn}F(n)\,dn=\frac{\pi}{4\cosh\{(1-i)\sqrt{(\alpha t)}\}\cosh\{(1+i)\sqrt{(\beta t)}\}}$$

in virtue of (1). Again, let

$$(4)\quad f(n)=-\frac{1}{2n}\sqrt{\left(\frac{\pi}{2n}\right)}$$
$$\times\Sigma\Sigma(-1)^{\frac{1}{2}(\mu+\nu)}\{\mu(1+i)\sqrt{\alpha}-\nu(1-i)\sqrt{\beta}\}\,e^{-(\pi\mu\nu-i\mu^2\alpha+i\nu^2\beta)/4n}$$
$$(\mu=1,\ 3,\ 5,\ \dots;\ \nu=1,\ 3,\ 5,\ \dots).$$

Then it is easy to show that

$$(5)\quad \int_0^\infty e^{-2tn}f(n)dn=\frac{\pi}{4\cosh\{(1-i)\sqrt{(\alpha t)}\}\cosh\{(1+i)\sqrt{(\beta t)}\}}.$$

Hence, by a theorem due to Lerch,* we obtain

$$(6)\qquad F(n)=f(n)$$

for all positive values of n, provided that $\alpha\beta=\frac{1}{4}\pi^2$. In particular, when $\alpha=\beta=\frac{1}{2}\pi$, we have

$$(7)\quad \frac{\sin\frac{1}{2}\pi n}{\cosh\frac{1}{2}\pi}-\frac{3\sin\frac{9}{2}\pi n}{\cosh\frac{3}{2}\pi}+\frac{5\sin\frac{25}{2}\pi n}{\cosh\frac{5}{2}\pi}-\dots$$
$$=-\frac{1}{4n\sqrt{n}}\Sigma\Sigma(-1)^{\frac{1}{2}(\mu+\nu)}e^{-\pi\mu\nu/4n}\left[(\mu+\nu)\cos\frac{\pi(\mu^2-\nu^2)}{4n}\right.$$
$$\left.+(\mu-\nu)\sin\frac{\pi(\mu^2-\nu^2)}{4n}\right]$$
$$(\mu=1,\ 3,\ 5,\ \dots;\ \nu=1,\ 3,\ 5,\ \dots)$$

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* See Mr. Hardy's note at the end of my previous paper.

for all positive values of n. As particular cases of (7), we have

$$(8)\quad \frac{\sin(\pi/a)}{\cosh\frac{1}{2}\pi} - \frac{3\sin(9\pi/a)}{\cosh\frac{3}{2}\pi} + \frac{5\sin(25\pi/a)}{\cosh\frac{5}{2}\pi} - \dots$$

$$= \frac{a\sqrt{a}}{8\sqrt{2}}\left\{\frac{1}{\cosh\frac{1}{8}\pi a} - \frac{3}{\cosh\frac{3}{8}\pi a} + \frac{5}{\cosh\frac{5}{8}\pi a} - \dots\right\}$$

$$= \frac{a\sqrt{a}}{4\sqrt{2}}\left\{e^{-\frac{1}{32}\pi a} - e^{-\frac{9}{32}\pi a} - e^{-\frac{25}{32}\pi a} + e^{-\frac{49}{32}\pi a} + \dots\right\}^4,$$

if a is a positive even integer; and

$$(9)\quad \frac{\sin(\pi/a)}{\cosh\frac{1}{2}\pi} - \frac{3\sin(9\pi/a)}{\cosh\frac{3}{2}\pi} + \frac{5\sin(25\pi/a)}{\cosh\frac{5}{2}\pi} - \dots$$

$$= \frac{a\sqrt{a}}{8\sqrt{2}}\left\{\frac{1}{\sinh\frac{1}{8}\pi a} + \frac{3}{\sinh\frac{3}{8}\pi a} + \frac{5}{\sinh\frac{5}{8}\pi a} + \dots\right\}$$

$$= \frac{a\sqrt{a}}{4\sqrt{2}}\left\{e^{-\frac{1}{32}\pi a} + e^{-\frac{9}{32}\pi a} + e^{-\frac{25}{32}\pi a} + e^{-\frac{49}{32}\pi a} + \dots\right\}^4,$$

if a is a positive odd integer; and so on.

3. It is also easy to show that if $\alpha\beta = \pi^2$, then

$$(10)\quad \left\{\frac{\alpha}{(\alpha+t)\sinh\alpha} - \frac{2\alpha}{(4\alpha+t)\sinh 2\alpha} + \frac{3\alpha}{(9\alpha+t)\sinh 3\alpha} - \dots\right\}$$

$$- \left\{\frac{\beta}{(\beta-t)\sinh\beta} - \frac{2\beta}{(4\beta-t)\sinh 2\beta} + \frac{3\beta}{(9\beta-t)\sinh 3\beta} - \dots\right\}$$

$$= \frac{1}{2t} - \frac{\pi}{2\sin\sqrt{(\alpha t)}\sinh\sqrt{(\beta t)}}.$$

From this we can deduce, as in the previous section, that if $\alpha\beta = \pi^2$, then

$$(11)\quad \frac{\alpha e^{in\alpha}}{\sinh\alpha} - \frac{2\alpha e^{4in\alpha}}{\sinh 2\alpha} + \frac{3\alpha e^{9in\alpha}}{\sinh 3\alpha} - \dots$$

$$+ \frac{\beta e^{-in\beta}}{\sinh\beta} - \frac{2\beta e^{-4in\beta}}{\sinh 2\beta} + \frac{3\beta e^{-9in\beta}}{\sinh 3\beta} - \dots$$

$$= \frac{1}{2} - \frac{1}{n}\sqrt{\left(\frac{\pi}{2n}\right)}$$

$$\times \Sigma\Sigma\left\{\mu(1-i)\sqrt{\alpha} + \nu(1+i)\sqrt{\beta}\right\} e^{-(2\pi\mu\nu - i\mu^2\alpha + i\nu^2\beta)/4n}$$

$$(\mu = 1, 3, 5, \dots;\ \nu = 1, 3, 5, \dots)$$

for all positive values of n. If, in particular, we put $\alpha=\beta=\pi$, we obtain

$$(12)\quad \frac{1}{4\pi}-\frac{\cos\pi n}{\sinh\pi}+\frac{2\cos 4\pi n}{\sinh 2\pi}-\frac{3\cos 9\pi n}{\sinh 3\pi}+\dots$$

$$=\frac{1}{2n\sqrt{(2n)}}\Sigma\Sigma\, e^{-\pi\mu\nu/2n}\left\{(\mu+\nu)\cos\frac{\pi(\mu^2+\nu^2)}{4n}+(\mu-\nu)\sin\frac{\pi(\mu^2+\nu^2)}{4n}\right\}$$

$$(\mu=1,\ 3,\ 5,\ \dots;\ \nu=1,\ 3,\ 5,\ \dots)$$

for all positive values of n. Thus, for example, we have

$$(13)\quad \frac{1}{4\pi}-\frac{\cos(2\pi/a)}{\sinh\pi}+\frac{2\cos(8\pi/a)}{\sinh 2\pi}-\frac{3\cos(18\pi/a)}{\sinh 3\pi}+\dots$$

$$=\tfrac{1}{8}a\sqrt{a}\left\{\frac{1}{\sinh\frac{1}{4}\pi a}+\frac{3}{\sinh\frac{3}{4}\pi a}+\frac{5}{\sinh\frac{5}{4}\pi a}+\dots\right\}$$

$$=\tfrac{1}{4}a\sqrt{a}\left\{e^{-\frac{1}{16}\pi a}+e^{-\frac{9}{16}\pi a}+e^{-\frac{25}{16}\pi a}+e^{-\frac{49}{16}\pi a}+\dots\right\}^4,$$

if a is a positive even integer; and

$$(14)\quad \frac{1}{4\pi}-\frac{\cos(2\pi/a)}{\sinh\pi}+\frac{2\cos(8\pi/a)}{\sinh 2\pi}-\frac{3\cos(18\pi/a)}{\sinh 3\pi}+\dots$$

$$=\tfrac{1}{8}a\sqrt{a}\left\{\frac{1}{\cosh\frac{1}{4}\pi a}-\frac{3}{\cosh\frac{3}{4}\pi a}+\frac{5}{\cosh\frac{5}{4}\pi a}-\dots\right\}$$

$$=\tfrac{1}{4}a\sqrt{a}\left\{e^{-\frac{1}{16}\pi a}-e^{-\frac{9}{16}\pi a}-e^{-\frac{25}{16}\pi a}+e^{-\frac{49}{16}\pi a}+\dots\right\}^4,$$

if a is a positive odd integer.

4. In a similar manner we can show that, if $\alpha\beta=\pi^2$, then

$$(15)\quad \frac{\alpha e^{in\alpha}}{e^{2\alpha}-1}+\frac{2\alpha e^{4in\alpha}}{e^{4\alpha}-1}+\frac{3\alpha e^{9in\alpha}}{e^{6\alpha}-1}+\dots$$

$$+\frac{\beta e^{-in\beta}}{e^{2\beta}-1}+\frac{2\beta e^{-4in\beta}}{e^{4\beta}-1}+\frac{3\beta e^{-9in\beta}}{e^{6\beta}-1}+\dots$$

$$=\alpha\int_0^\infty\frac{xe^{-in\alpha x^2}}{e^{2\pi x}-1}\,dx^*+\beta\int_0^\infty\frac{xe^{in\beta x^2}}{e^{2\pi x}-1}\,dx^*-\tfrac{1}{4}$$

$$+\frac{1}{n}\sqrt{\left(\frac{\pi}{2n}\right)}\sum_{\mu=1}^{\mu=\infty}\sum_{\nu=1}^{\nu=\infty}\{\mu(1-i)\sqrt{\alpha}+\nu(1+i)\sqrt{\beta}\}\,e^{-(2\pi\mu\nu-i\mu^2\alpha+i\nu^2\beta)/n}$$

* I showed in my former paper that this integral can be calculated in finite terms whenever $n\alpha$ is a rational multiple of π. I take this opportunity of correcting a mistake: in the formulæ (48) the first integral is $\frac{1}{12}$ and not $\frac{1}{16}$.

for all positive values of n. Putting $\alpha=\beta=\pi$ in (15) we see that, if $n>0$, then

$$(16)\quad \frac{1}{8\pi}+\frac{\cos\pi n}{e^{2\pi}-1}+\frac{2\cos 4\pi n}{e^{4\pi}-1}+\frac{3\cos 9\pi n}{e^{6\pi}-1}+\dots$$

$$=\int_0^\infty \frac{x\cos\pi n x^2}{e^{2\pi x}-1}\,dx+\frac{1}{2n\sqrt{(2n)}}\sum_{\mu=1}^{\mu=\infty}\sum_{\nu=1}^{\nu=\infty} e^{-2\pi\mu\nu/n}$$

$$\times\left[(\mu+\nu)\cos\left\{\frac{\pi(\mu^2-\nu^2)}{n}\right\}+(\mu-\nu)\sin\left\{\frac{\pi(\mu^2-\nu^2)}{n}\right\}\right].$$

As particular cases of (16) we have

$$(17)\quad \frac{1}{8\pi}+\frac{\cos(\pi/a)}{e^{2\pi}-1}+\frac{2\cos(4\pi/a)}{e^{4\pi}-1}+\frac{3\cos(9\pi/a)}{e^{6\pi}-1}+\dots$$

$$=\int_0^\infty \frac{x\cos(\pi x^2/a)}{e^{2\pi x}-1}\,dx+a\sqrt{(\tfrac{1}{2}a)}\left(\frac{1}{e^{2\pi a}-1}+\frac{2}{e^{4\pi a}-1}+\frac{3}{e^{6\pi a}-1}+\dots\right),$$

if a is a positive even integer;

$$(18)\quad \frac{1}{8\pi}+\frac{\cos(\pi/a)}{e^{2\pi}-1}+\frac{2\cos(4\pi/a)}{e^{4\pi}-1}+\frac{3\cos(9\pi/a)}{e^{6\pi}-1}+\dots$$

$$=\int_0^\infty \frac{x\cos(\pi x^2/a)}{e^{2\pi x}-1}\,dx+a\sqrt{(\tfrac{1}{2}a)}\left(\frac{1}{e^{2\pi a}+1}-\frac{2}{e^{4\pi a}+1}+\frac{3}{e^{6\pi a}+1}-\dots\right),$$

if a is a positive odd integer; and

$$(19)\quad \frac{1}{8\pi}+\frac{\cos(2\pi/a)}{e^{2\pi}-1}+\frac{2\cos(8\pi/a)}{e^{4\pi}-1}+\frac{3\cos(18\pi/a)}{e^{6\pi}-1}+\dots$$

$$=\int_0^\infty \frac{x\cos(2\pi x^2/a)}{e^{2\pi x}-1}\,dx+\tfrac{1}{4}a\sqrt{a}\left(\frac{1}{e^{\pi a}+1}+\frac{3}{e^{3\pi a}+1}+\frac{5}{e^{5\pi a}+1}+\dots\right),$$

if a is a positive odd integer.

It may be interesting to note that different functions dealt with in this paper have the same asymptotic expansion for small values of n. For example, the two different functions

$$\frac{1}{8\pi}+\frac{\cos n}{e^{2\pi}-1}+\frac{2\cos 4n}{e^{4\pi}-1}+\frac{3\cos 9n}{e^{6\pi}-1}+\dots$$

and

$$\int_0^\infty \frac{x\cos nx^2}{e^{2\pi x}-1}\,dx$$

have the same asymptotic expansion, viz.

$$(20)\quad \frac{1}{24}-\frac{n^2}{1008}+\frac{n^4}{6336}-\frac{n^6}{17280}+\dots .^*$$

* This series (in spite of the appearance of the first few terms) diverges for all values of n.

THE EXPANSION OF THE SQUARE OF A BESSEL FUNCTION IN THE FORM OF A SERIES OF BESSEL FUNCTIONS.

By *A. E. Jolliffe, M.A.*

THE square of the Bessel function $J_n(x)$ can be expanded in a series of Bessel functions with the argument $2x$ in the form

$$a_0\{J_{2n}(2x)+J_{2n+2}(2x)\}+a_2\{J_{2n+4}(2x)+J_{2n+6}(2x)\}+\dots$$
$$+a_{2r}\{J_{2n+4r}(2x)+J_{2n+4r+2}(2x)\}+\dots,$$

where
$$a_0=\frac{(2n)!}{(n!)^2 2^{2n}},$$

and
$$\frac{a_{2r}}{a_{2r-2}}=\frac{(2r-1)(2n+2r-1)}{2r(2n+2r)}.$$

(So far as I can discover, this expansion has not been given before).

$\{J_n(x)\}^2$ satisfies the differential equation

$$\frac{d}{d\xi}\left\{\xi^2\frac{d^2y}{d\xi^2}+\xi\frac{dy}{d\xi}+(\tfrac{1}{2}\xi^2-2n^2)y\right\}+(\tfrac{1}{2}\xi^2-2n^2)\frac{dy}{d\xi}=0,$$

where ξ denotes $2x$.

If we write $J_r(\xi)$ for y in the left-hand side, we obtain

$$(r^2-4n^2)\,J_r'(\xi)-\xi J_r(\xi).$$

By means of the formulæ

$$2J_r'(\xi)=J_{r-1}(\xi)-J_{r+1}(\xi),$$
$$2rJ_r(\xi)=\xi\{J_{r-1}(\xi)+J_{r+1}(\xi)\},$$

the result of substituting

$$a_0\{J_{2n}(\xi)+J_{2n+2}(\xi)\}+a_2\{J_{2n+4}(\xi)+J_{2n+6}(\xi)\}+\dots$$

in the left-hand side of the differential equation can be reduced to

$$2a_0\{-1(2n+1)J_{2n+3}\}+2a_2\{2(2n+2)J_{2n+3}-3(2n+3)J_{2n+7}\}$$
$$+2a_4\{4(2n+4)J_{2n+7}-5(2n+5)J_{2n+11}\}+\dots,$$

which shows that, when

$$\frac{a_{2r}}{a_{2r-2}}=\frac{(2r-1)(2n+2r-1)}{2r(2n+2r)},$$

the series is a solution of the differential equation. When a_0 is properly determined, it must be the square of $J_n(x)$. The value of a_0 is determined as that given above, by considering the coefficient of x^{2n} in the expansions of the series and $\{J_n(x)\}^2$ in powers of x.

SOME PROPERTIES OF THE TETRAHEDRON AND ITS SPHERES.

By *T. C. Lewis, M.A.*

1. So far as I am able to ascertain, no direct elementary proof of the following proposition has hitherto appeared.

If A' be the vertex and BCD the base of any tetrahedron, then the sphere which passes through the points B, C, D and touches the inscribed sphere of the tetrahedron will also touch the sphere escribed on the base.

This may be proved by means of the penta-spherical coordinates of Gaston Darboux.

Take as a tetrahedron of reference any orthocentric tetrahedron $ABCD$ on the same base BCD.

The equations of the faces of the tetrahedron $A'BCD$ are

$$\left.\begin{aligned} \rho_1 x_1 - \rho_5 x_5 &= 0 \\ \rho_2 x_2 - \rho_5 x_5 &= k_2(\rho_1 x_1 - \rho_5 x_5) \\ \rho_3 x_3 - \rho_5 x_5 &= k_3(\rho_1 x_1 - \rho_5 x_5) \\ \rho_4 x_4 - \rho_5 x_5 &= k_4(\rho_1 x_1 - \rho_5 x_5) \end{aligned}\right\} \dots\dots\dots\dots\dots \text{(i)}.$$

Let $\Sigma \alpha_k x_k = 0$ be the equation to the inscribed sphere, of radius r. Let the distances of A, B, C, D from the orthocentre be a_1, a_2, a_3, a_4. Then

$$\begin{aligned} \alpha_1\rho_1 - \alpha_5\rho_5 &= a_1, \\ \alpha_2\rho_2 - \alpha_5\rho_5 &= k_2 a_1 + \sqrt{\{k_2^2\rho_1^2 + \rho_2^2 + (1-k_2)^2\rho_5^2\}} \\ &= k_2 a_1 + K_2, \\ \alpha_3\rho_3 - \alpha_5\rho_5 &= k_3 a_1 + K_3, \\ \alpha_4\rho_4 - \alpha_5\rho_5 &= k_4 a_1 + K_4, \end{aligned}$$

where K_2, K_3, K_4 are the values of $\sqrt{\{k_n^2\rho_1^2 + \rho_n^2 + (1-k_n)^2\rho_5^2\}}$ or $\sqrt{\{k_n^2 a_1^2 + a_n^2 - 2k_n\rho_5^2\}}$, when n is 2, 3, or 4 respectively.

Therefore $\Sigma\alpha^2 = 1$

$$= \frac{1}{\rho_1^2}(\alpha_5\rho_5 + a_1)^2 + \frac{1}{\rho_2^2}\{\alpha_5\rho_5 + k_2 a_1 + K_2\}^2 + \dots + \frac{1}{\rho_5^2}\alpha_5^2\rho_5^2$$

$$= 2\alpha_5\rho_5\left\{\left(\frac{1}{\rho_1^2} + \frac{k_2}{\rho_2^2} + \frac{k_3}{\rho_3^2} + \frac{k_4}{\rho_4^2}\right)a_1 + \frac{K_2}{\rho_2^2} + \frac{K_3}{\rho_3^2} + \frac{K_4}{\rho_4^2}\right\}$$

$$+ 2a_1\left(\frac{k_2K_2}{\rho_2^2} + \frac{k_3K_3}{\rho_3^2} + \frac{k_4K_4}{\rho_4^2}\right)$$

$$+ a_1^2\left(\frac{k_2^2}{\rho_2^2} + \frac{k_3^2}{\rho_3^2} + \frac{k_4^2}{\rho_4^2}\right) + 3 - 2\rho_5^2\left(\frac{k_2}{\rho_2^2} + \frac{k_3}{\rho_3^2} + \frac{k_4}{\rho_4^2}\right),$$

therefore

$$\alpha_5 \rho_5 \left\{\left(\frac{1}{\rho_1^2} + \frac{k_2}{\rho_2^2} + \frac{k_3}{\rho_3^2} + \frac{k_4}{\rho_4^2}\right) a_1 + \frac{K_2}{\rho_2^2} + \frac{K_3}{\rho_3^2} + \frac{K_4}{\rho_4^2}\right\}$$
$$+ 1 - \rho_5^2 \left(\frac{k_2}{\rho_2^2} + \frac{k_3}{\rho_3^2} + \frac{k_4}{\rho_4^2}\right) + a_1^2 \left(\frac{k_2^2}{\rho_2^2} + \frac{k_3^2}{\rho_3^2} + \frac{k_4^2}{\rho_4^2}\right)$$
$$+ a_1 \left\{\frac{k_2 K_2}{\rho_2^2} + \frac{k_3 K_3}{\rho_3^2} + \frac{k_4 K_4}{\rho_4^2}\right\} = 0 \ldots\ldots\ldots\ldots \text{(ii)}.$$

Here the coefficient of $\alpha_5 \rho_5$ is seen to be $1/r$; let H be the sum of the terms independent of α_5.

$$\left.\begin{array}{ll} \text{Then} & 1/r \,.\, \alpha_5 \rho_5 + H = 0, \\ \text{Thus} & \alpha_1 \rho_1 = a_1 - Hr \\ & \alpha_2 \rho_2 = k_2 a_1 - Hr + K_2 \\ & \&\text{c.} = \&\text{c.} \end{array}\right\} \ldots\ldots\ldots\ldots\ldots \text{(iii)}.$$

The inscribed sphere $\Sigma \alpha_k x_k = 0$ is therefore determined.

2. Let a sphere pass through the points B, C, D. Its equation will be

$$-m\rho_1 x_1 + \rho_2 x_2 + \rho_3 x_3 + \rho_4 x_4 + (m-1)\,\rho_5 x_5 = 0 \ldots\ldots \text{(iv)}.$$

Its radius R' is given by

$$4R'^2 = m^2 a_1^2 - 2m\rho_5^2 + \rho_2^2 + \rho_3^2 + \rho_4^2 + \rho_5^2 \ldots\ldots\ldots \text{(v)}.$$

Let this sphere touch the inscribed sphere whose equation has already been determined; then the following condition is satisfied, viz.

$$-m\alpha_1\rho_1 + \alpha_2\rho_2 + \alpha_3\rho_3 + \alpha_4\rho_4 + (m-1)\,\alpha_5\rho_5$$
$$= \sqrt{\{m^2 a_1^2 - 2m\,\rho_5^2 + \rho_2^2 + \rho_3^2 + \rho_4^2 + \rho_5^2\}},$$

therefore

$$\{-ma_1 + (k_2 + k_3 + k_4)\, a_1 - 2Hr + K_2 + K_3 + K_4\}^2$$
$$= m^2 a_1^2 - 2m\rho_5^2 + \rho_2^2 + \rho_3^2 + \rho_4^2 + \rho_5^2,$$

therefore

$$-2ma_1 \left\{(k_2 + k_3 + k_4)\, a_1 - 2Hr + K_2 + K_3 + K_4 - \frac{\rho_5^2}{a_1}\right\}$$
$$= \rho_2^2 + \rho_3^2 + \rho_4^2 + \rho_5^2 - (k_2 + k_3 + k_4)^2 a_1^2 - 4H^2 r^2$$
$$- (k_2^2 + k_3^2 + k_4^2)\, a_1^2 - \rho_2^2 - \rho_3^2 - \rho_4^2 - 3\rho_5^2$$
$$+ 2\,(k_2 + k_3 + k_4)\,\rho_5^2 + 4Hr\,(k_2 + k_3 + k_4)\, a_1$$
$$- 2\,(k_2 + k_3 + k_4)\, a_1\, (K_2 + K_3 + K_4)$$
$$+ 4Hr\,(K_2 + K_3 + K_4) - 2K_3 K_4 - 2K_4 K_2 - 2K_2 K_3.$$

Therefore

$$ma_1\left\{(k_2+k_3+k_4)\,a_1-2Hr+K_2+K_3+K_4-\frac{\rho_5^{\,2}}{a_1}\right\}$$

$$=\rho_5^{\,2}+(k_2^{\,2}+k_3^{\,2}+k_4^{\,2}+k_3k_4+k_4k_2+k_2k_3)\,a_1^{\,2}+2H^2r^2-(k_2+k_3+k_4)\,\rho_5^{\,2}$$

$$-2Hr(k_2+k_3+k_4)\,a_1+(k_2+k_3+k_4)\,a_1(K_2+K_3+K_4)$$

$$-2Hr\,(K_2+K_3+K_4)+K_3K_4+K_4K_2+K_2K_3\ldots\text{(vi)}.$$

3. This determines m in order that the sphere through B, C, D may touch the inscribed sphere. If it also touches the sphere escribed on the base, whose radius is r_1, this equation must remain true if r is changed to r_1, the sign of a_1 being changed, and the corresponding value of H being H_1. The necessary condition is that, whatever values be given to k_2, k_3, k_4,

$$2HH_1+\left\{K_3K_4+K_4K_2+K_2K_3-(k_3k_4+k_4k_2+k_2k_3)\,a_1^{\,2}+\rho_5^{\,2}\right\}\frac{1}{rr_1}$$

$$+\left(\frac{H}{r_1}-\frac{H_1}{r}\right)\left\{(k_2+k_3+k_4)\,a_1-\frac{\rho_5^{\,2}}{a_1}\right\}$$

$$=\left(\frac{H}{r_1}+\frac{H_1}{r}\right)(K_2+K_3+K_4)\ldots\text{(vii)}.$$

If the known expressions for $\frac{1}{r}$, $\frac{1}{r_1}$, H, H_1 are substituted, the above identity may be demonstrated by a piece of work which, though lengthy, presents no mathematical difficulty.

4. By making use of this identity the equation to determine m so that the sphere (iv) may touch the inscribed sphere reduces to

$$\frac{m}{rr_1}=-\frac{1}{a_1}\left(\frac{H}{r_1}-\frac{H_1}{r}\right)+(k_2+k_3+k_4)\frac{1}{rr_1}\;\ldots\ldots\text{(viii)},$$

which remains the same when r and r_1, H and H_1 are interchanged, and the sign of a_1 is also changed.

Therefore the proposition is proved.

5. But as a matter of fact we have proved more than we set out to prove, namely the following comprehensive theorem:

Through the angular points of any face of a tetrahedron there may be drawn four spheres each of which touches two of the eight spheres which touch all the four faces of the tetrahedron.

In the above work if K_2 be negative instead of positive, the place of the inscribed sphere therein is taken by the sphere escribed on the face $A'CD$, whose radius is r_2; at the same time let H become H_2. The identity (vii) remains true, *mutatis mutandis.* Thus we obtain a sphere which passes through B, C, D, and touches the sphere escribed on $A'CD$ and also touches the sphere which touches $A'CD$ and BCD on the reverse side, *i.e.* on the opposite side to that on which the inscribed sphere touches them—if there is such a second sphere. But if there is not a sphere touching the planes opposite A' and B on the reverse side, there will be one touching the planes opposite C and D on the reverse side, and in determining it K_2, K_3, K_4 will be opposite in sign from what they are for the sphere escribed opposite B.

Now if m in (viii) be evaluated we obtain

$$\frac{m}{rr_1} = \frac{2k_2K_3K_4}{\rho_3^2\rho_4^2} + \frac{2k_3K_4K_2}{\rho_4^2\rho_2^2} + \frac{2k_4K_2K_3}{\rho_2^2\rho_3^2}$$

+ terms independent of the sign of K_2, K_3, K_4.

Therefore m also remains unaltered, not only if the sign of a_1 is changed, but also if the signs of K_2, K_3, or K_4 are *all* changed; and this whatever the signs of K_2, K_3, or K_4 may be at first, the corresponding values of r and H being taken.

Thus the general theorem is established.

6. If $A'BCD$ is an orthocentric tetrahedron A' must lie on the line

$$\rho_2x_2 = \rho_3x_3 = \rho_4x_4,$$

and therefore it is clear from the equations (i) that the necessary condition is

$$k_2 = k_3 = k_4.$$

But we may take this orthocentric tetrahedron as the tetrahedron of reference, so that $k_2 = k_3 = k_4 = 0$, and $H = H_1 = 1$. Also $K_2 = a_2$, $K_3 = a_3$, $K_4 = a_4$.

Therefore the identity (vii) becomes

$$2 + (a_3a_4 + a_4a_2 + a_2a_3 + \rho_5^2)\frac{1}{rr_1} + \left(\frac{1}{r} - \frac{1}{r_1}\right)\frac{\rho_5^2}{a_1}$$

$$= (a_2 + a_3 + a_4)\left(\frac{1}{r} + \frac{1}{r_1}\right),$$

which reduces to

$$a_3a_4 + a_4a_2 + a_2a_3 + \rho_5^2 - (a_1 + a_2 + a_3 + a_4)(r_1 + r) + 2a_1r_1 = 0,$$

and there are three other similar identities corresponding to the different vertices, therefore by addition the following symmetrical identity is found:—

$$
\begin{aligned}
2\,(a_1a_2 + a_1a_3 + a_1a_4 + a_2a_3 + a_2a_4 + a_3a_4)& \\
+ 4\rho_5^{\,2} + 2\,(a_1r_1 + a_2r_2 + a_3r_3 + a_4r_4)& \\
= (a_1 + a_2 + a_3 + a_4)\,(4r + r_1 + r_2 + r_3 + r_4)^{*} \ldots (\text{ix}).&
\end{aligned}
$$

DETERMINANTS WHOSE ELEMENTS ARE ALTERNATING NUMBERS.

By *Thomas Muir, LL.D.*

1. In the *Proceedings of the London Mathematical Society*, VII. (1876), pp. 100—112, there is a paper by Spottiswoode on alternating numbers viewed as the elements of a determinant. The paper contains a fairly large number of results, particular and general, some of which were, he says, obtained from notes by Clifford. Probably the most noteworthy theorem in the collection and at the same time the least satisfactory is that on multiplication, and to it I wish to direct a little attention.

2. Spottiswoode says (p. 103): "the ordinary formula for the multiplication of determinants may be applied, namely,

$$
\begin{array}{llll}
-\,\lambda,\ \mu,\ \ldots & \lambda',\ \mu',\ \ldots & = & (\lambda\lambda'),\ (\lambda\mu'),\ \ldots, \\
\ \ 1,\ 2,\ \ldots & 1,\ 2,\ \ldots & & (\mu\lambda'),\ (\mu\mu'),\ \ldots, \\
 & & & \ldots\ldots\ldots\ldots\ldots\ldots,
\end{array}
$$

if it be understood that, after developing the right-hand side of the equation according to the ordinary rule, those terms which require an odd number of changes to bring them into

---

* If $a_4 = \infty$, the following result with regard to a plane triangle is deduced, viz.

$$
\begin{aligned}
2\,(a_1 + a_2 + a_3) &= r_1 + r_2 + r_3 + 3r \\
&= 4R + 4r,
\end{aligned}
$$

i.e. the sum of the distances of the vertices of a triangle from the orthocentre is equal to the sum of the diameters of the inscribed and circumscribed circles, a property which may be easily proved by elementary trigonometry. Any distance a_1 is negative if drawn from an obtuse angle of the triangle.

If r_a, r_b, r_c are the radii of the circles inscribed in PBC, PCA, PAB, respectively, the sum of the sides of an acute-angled triangle may be expressed thus—

$$
\begin{aligned}
a + b + c &= 4R + r + r_a + r_b + r_c \\
&= r_1 + r_2 + r_3 + r_a + r_b + r_c.
\end{aligned}
$$

If C is an obtuse angle, $\quad a + b - c = r + r_c - r_a - r_b.$

These properties of the triangle are appended because the author has not met with them before.

the form $\lambda\mu\ldots\lambda'\mu'\ldots$ are to have their signs altered, while those which require an even number of changes are to retain their signs."

In regard to this we may note first that the formula can be expressed much more simply. The determinant on its right is, as conditioned by him, no determinant at all, but a function differing from a determinant in having all its terms positive, and usually known as a permanent with the notation $\overset{+}{|}\ \overset{+}{|}$. His theorem thus is for the third order

$$\begin{vmatrix} \lambda_1 & \lambda_2 & \lambda_3 \\ \mu_1 & \mu_2 & \mu_3 \\ \nu_1 & \nu_2 & \nu_3 \end{vmatrix} \cdot \begin{vmatrix} \alpha_1 & \alpha_2 & \alpha_3 \\ \beta_1 & \beta_2 & \beta_3 \\ \gamma_1 & \gamma_2 & \gamma_3 \end{vmatrix} = - \overset{+}{|} \begin{matrix} \Sigma\lambda\alpha & \Sigma\lambda\beta & \Sigma\lambda\gamma \\ \Sigma\mu\alpha & \Sigma\mu\beta & \Sigma\mu\gamma \\ \Sigma\nu\alpha & \Sigma\nu\beta & \Sigma\nu\gamma \end{matrix} \overset{+}{|} .$$

In the second place it has to be noted that what Spottiswoode means by calling the elements of $|\lambda_1\mu_2\nu_3|$, $|\alpha_1\beta_2\gamma_3|$ alternating numbers is simply that any two of the eighteen, λ_2 and γ_1 say, are such that

$$\lambda_2\gamma_1 = -\gamma_1\lambda_2.$$

3. Save for a verification of his formula in the case where the determinants are of the second order, there is no demonstration given. Probably he verified it also for the third order; but further than the third he could not have gone, otherwise he would have seen that the sign preceding the determinant on the right is not always minus. Thus, taking the case of the fourth order, namely,

$$|\lambda_1\mu_2\nu_3\rho_4| \cdot |\alpha_1\beta_2\gamma_3\delta_4| = - \overset{+}{|} \Sigma\lambda\alpha \,.\, \Sigma\mu\beta \,.\, \Sigma\nu\gamma \,.\, \Sigma\rho\delta \overset{+}{|},$$

we see that the product of the two diagonal terms on the left is

$$\lambda_1\mu_2\nu_3\rho_4 \,.\, \alpha_1\beta_2\gamma_3\delta_4,$$

and that the diagonal term on the right is

$$- \Sigma\lambda\alpha \,.\, \Sigma\mu\beta \,.\, \Sigma\nu\gamma \,.\, \Sigma\rho\delta.$$

This latter being

$$-(\lambda_1\alpha_1 + \ldots + \lambda_4\alpha_4)(\mu_1\beta_1 + \ldots + \mu_4\beta_4)(\nu_1\gamma_1 + \ldots + \nu_4\gamma_4)(\rho_1\delta_1 + \ldots + \rho_4\delta_4),$$

one of the terms of its expansion is

$$-\lambda_1\alpha_1 \,.\, \mu_2\beta_2 \,.\, \nu_3\gamma_3 \,.\, \rho_4\delta_4,$$

which on the shifting forward of μ_2, ν_3, ρ_4 becomes

$$-(-1)^{1+2+3}\lambda_1\mu_2\nu_3\rho_4 \,.\, \alpha_1\beta_2\gamma_3\delta_4.$$

Thus the minus sign preceding the permanent cannot be correct.

4. Let us therefore examine the matter anew.

When the elements are ordinary numbers we know that

$$|\lambda_1\mu_2\nu_3\rho_4| \cdot |\alpha_1\beta_2\gamma_3\delta_4| = \begin{vmatrix} \Sigma\lambda\alpha & \Sigma\lambda\beta & \Sigma\lambda\gamma & \Sigma\lambda\delta \\ \Sigma\mu\alpha & \Sigma\mu\beta & \Sigma\mu\gamma & \Sigma\mu\delta \\ \Sigma\nu\alpha & \Sigma\nu\beta & \Sigma\nu\gamma & \Sigma\nu\delta \\ \Sigma\rho\alpha & \Sigma\rho\beta & \Sigma\rho\gamma & \Sigma\rho\delta \end{vmatrix}$$

and that this comes about because every term on the left is matched by an equal term on the right, and because all the remaining terms on the right are cancellable in pairs. For example, if we take any term of the determinant on the right, say

$$(-1)^{\omega}\, \Sigma\lambda\beta \,.\, \Sigma\mu\delta \,.\, \Sigma\nu\gamma \,.\, \Sigma\rho\alpha,$$

where ω is the number of inverted-pairs in $\beta\delta\gamma\alpha$, and select from the 4^4 sub-terms involved in it the sub-term

$$(-1)^{\omega} . \lambda_a\beta_a \cdot \mu_b\delta_b \cdot \nu_c\gamma_c \cdot \rho_d\alpha_d, \qquad (R)$$

this sub-term is either matched on the left-hand side by the term

$$(-1)^{\omega} . \lambda_a\mu_b\nu_c\rho_d \cdot \alpha_d\beta_a\gamma_c\delta_b, \qquad (L)$$

in which case a, b, c, d is a permutation of 1, 2, 3, 4; or, it is one of the many cancellable sub-terms having a, b, c, d not all different.

With this before us let us now consider the effect of changing the elements from ordinary into alternating numbers. Taking first the case where a, b, c, d are all different, we readily see that identity is no longer ensured by the two terms R and L being merely composed of the same factors: the *order* in which the factors appear must be the same as well. We must be able therefore in accordance with the laws of alternating numbers to alter as required the order of R's factors without bringing about a sign-factor differing from that of L. Now, in order that the factors λ_a, μ_b, ν_c, ρ_d in R may be made the first four of the eight, the number of sign-changes necessary is

$$1 + 2 + 3,$$

and in order that the remaining factors β_a, δ_b, γ_c, α_d may appear in the order $\alpha_d\beta_a\gamma_c\delta_b$ the number of sign-changes necessary is

$$\omega.$$

Consequently, the index of R's sign-factor after being increased by $1+2+3+\omega$ must still be equivalent to the index of L's sign-factor—a manifest impossibility so long as R's original sign-factor is reckoned according to the sign-law of determinants. We observe, however, that if we make the original sign-factor of R not $(-1)^{\omega}$ but $(-1)^{1+2+3}$ the requisite transposition of factors would change it into

$$(-1)^{1+2+3+1+2+3+\omega}, \quad i.e. \ (-1)^{\omega},$$

as desired. It is thus suggested that for alternating elements all the signs on the right should be the same, namely, $(-1)^{1+2+3}$; in other words, that the right-hand side of the multiplication-identity should be changed from

$$|\Sigma\lambda\alpha . \Sigma\mu\beta . \Sigma\nu\gamma . \Sigma\rho\delta| \quad \text{into} \quad (-1)^{1+2+3}\overset{+}{|}\Sigma\lambda\alpha . \Sigma\mu\beta . \Sigma\nu\gamma . \Sigma\rho\delta\overset{+}{|}.$$

In order fully to justify the change it is of course necessary to consider the other case, namely, where a, b, c, d are not all different. When this is done, however, it is found that cancellation takes place on the right exactly as before, the requisite difference of sign, which does not exist in a permanent to start with, being provided by the law of transposition of alternating numbers.

5. A more direct and more generally satisfactory way of establishing the identity, now that its true form is known, is by beginning with the permanent on the right and deducing from it the factors on the left. To this end the more elementary properties of permanents require to be known; namely, *The numbers employed being alternating numbers*

(a) *the interchange of two columns of a permanent does not alter its value;*

(b) *the interchange of two rows of a permanent alters only its sign;*

(c) *if two rows be alike, the permanent vanishes;*

(d) *if any two columns of a permanent be of the form*

$$\begin{array}{cc} \xi_1\alpha & \xi_1\beta \\ \xi_2\alpha & \xi_2\beta \\ \xi_3\alpha & \xi_3\beta \\ \vdots & \vdots \end{array}$$

it vanishes;

(e) *any determinant is expressible as a permanent differing only in having two rows interchanged;*

(f) *if the* p^{th} *row of an* n-*line permanent be multiplied by* ω, *the permanent is thereby multiplied by* $(-1)^{n-p}\,\omega$;

(g) *if the rows of an* n-*line permanent be multiplied in order by* ω_1, ω_2, ω_3, ... *respectively, the permanent is thereby multiplied by* $(-1)^{\frac{1}{2}n(n-1)}\,\omega_1\omega_2\omega_3\ldots$;

(h) *if the columns of an* n-*line permanent be multiplied in order by* ω_1, ω_2, ω_3, ... *respectively, the result is equal to the corresponding determinant multiplied by* $(-1)^{\frac{1}{2}n(n-1)}\,\omega_1\omega_2\omega_3\ldots$.

These are readily established in every case by considering the individual terms of the permanent or determinant concerned, and always bearing in mind that in the formation of the terms the elements are taken from the rows in order. For example, in the case of (d) it is sufficient to note that every term

$$\ldots\xi_r\alpha\ldots\xi_s\beta\ldots$$

is matched by another

$$\ldots\xi_r\beta\ldots\xi_s\alpha\ldots,$$

and that the two have necessarily different signs on account of α and β being alternating numbers. Again, in the case of (h) the given permanent being

$$\overset{+}{|}\alpha_1\beta_2\gamma_3\delta_4\ldots\overset{+}{|}\,,$$

we have to note that any term of the permanent resulting after the specified multiplication is of the form

$$\alpha_r\omega_r\,.\,\beta_s\omega_s\,.\,\gamma_t\omega_t\,.\,\delta_u\omega_u\ldots,$$

where $r, s, t, u, \ldots$ are the numbers of the columns from which the elements constituting the terms are taken. Now, by the transformation-law of alternating numbers, this is changeable, first, into

$$(-1)^{\frac{1}{2}n(n-1)}.\,\alpha_r\beta_s\gamma_t\delta_u\ldots\,.\,\omega_r\omega_s\omega_t\omega_u\ldots,$$

and thereafter into

$$(-1)^{\frac{1}{2}n(n-1)}.\,(-1)^{\pi}.\,\alpha_r\beta_s\gamma_t\delta_u\ldots\,.\,\omega_1\omega_2\omega_3\omega_4\ldots,$$

where π is the number of inverted-pairs in $r, s, t, u, \ldots$. But $(-1)^{\pi}\alpha_r\beta_s\gamma_t\delta_u\ldots$ is a term of the determinant $|\alpha_1\beta_2\gamma_3\delta_4\ldots|$; hence the result is

$$(-1)^{\frac{1}{2}n(n-1)}\,|\alpha_1\beta_2\gamma_3\delta_4\ldots|\,\omega_1\omega_2\omega_3\omega_4\ldots,$$

as affirmed.

6. These preliminaries being settled let us now consider the permanent

$$
{}^{+}\!\begin{vmatrix}
\lambda_1\alpha_1+\lambda_2\alpha_2+\lambda_3\alpha_3 & \lambda_1\beta_1+\lambda_2\beta_2+\lambda_3\beta_3 & \lambda_1\gamma_1+\lambda_2\gamma_2+\lambda_3\gamma_3 \\
\mu_1\alpha_1+\mu_2\alpha_2+\mu_3\alpha_3 & \mu_1\beta_1+\mu_2\beta_2+\mu_3\beta_3 & \mu_1\gamma_1+\mu_2\gamma_2+\mu_3\gamma_3 \\
\nu_1\alpha_1+\nu_2\alpha_2+\nu_3\alpha_3 & \nu_1\beta_1+\nu_2\beta_2+\nu_3\beta_3 & \nu_1\gamma_1+\nu_2\gamma_2+\nu_3\gamma_3
\end{vmatrix}^{+},
$$

the restriction to the third order being made merely for convenience in writing. In the first place the permanent is expressible as a sum of 27 permanents with monomial elements; and, this being done, it is seen that 21 of them are of the type dealt with in theorem (d) of the preceding paragraph, and therefore vanish. There thus only remains for consideration the six-termed expression

$$
{}^{+}\!\begin{vmatrix}
\lambda_1\alpha_1 & \lambda_2\beta_2 & \lambda_3\gamma_3 \\
\mu_1\alpha_1 & \mu_2\beta_2 & \mu_3\gamma_3 \\
\nu_1\alpha_1 & \nu_2\beta_2 & \nu_3\gamma_3
\end{vmatrix}^{+}
+ {}^{+}\!\begin{vmatrix}
\lambda_2\alpha_2 & \lambda_1\beta_1 & \lambda_3\gamma_3 \\
\mu_2\alpha_2 & \mu_1\beta_1 & \lambda_3\gamma_3 \\
\nu_2\alpha_2 & \nu_1\beta_1 & \nu_3\gamma_3
\end{vmatrix}^{+}
+ {}^{+}\!\begin{vmatrix}
\lambda_3\alpha_3 & \lambda_1\beta_1 & \lambda_2\gamma_2 \\
\mu_3\alpha_3 & \mu_1\beta_1 & \mu_2\gamma_2 \\
\nu_3\alpha_3 & \nu_1\beta_1 & \nu_2\gamma_2
\end{vmatrix}^{+}
$$

$$
+ {}^{+}\!\begin{vmatrix}
\lambda_1\alpha_1 & \lambda_3\beta_3 & \lambda_2\gamma_2 \\
\mu_1\alpha_1 & \mu_3\beta_3 & \mu_2\gamma_2 \\
\nu_1\alpha_1 & \nu_3\beta_3 & \nu_2\gamma_2
\end{vmatrix}^{+}
+ {}^{+}\!\begin{vmatrix}
\lambda_2\alpha_2 & \lambda_3\beta_3 & \lambda_1\gamma_1 \\
\mu_2\alpha_2 & \mu_3\beta_3 & \mu_1\gamma_1 \\
\nu_2\alpha_2 & \nu_3\beta_3 & \nu_1\gamma_1
\end{vmatrix}^{+}
+ {}^{+}\!\begin{vmatrix}
\lambda_3\alpha_3 & \lambda_2\beta_2 & \lambda_1\gamma_1 \\
\mu_3\alpha_3 & \mu_2\beta_2 & \mu_1\gamma_1 \\
\nu_3\alpha_3 & \nu_2\beta_2 & \nu_1\gamma_1
\end{vmatrix}^{+},
$$

which, by theorem (h) of § 5, is equal to

$$
(-1)^3 \left\{ \begin{array}{l}
\quad |\lambda_1\mu_2\nu_3| . \alpha_1\beta_2\gamma_3 + |\lambda_2\mu_1\nu_3| . \alpha_2\beta_1\gamma_3 + |\lambda_3\mu_1\nu_2| . \alpha_3\beta_1\gamma_2 \\
+ |\lambda_1\mu_3\nu_2| . \alpha_1\beta_3\gamma_2 + |\lambda_2\mu_3\nu_1| . \alpha_2\beta_3\gamma_1 + |\lambda_3\mu_2\nu_1| . \alpha_3\beta_2\gamma_1
\end{array} \right\},
$$

and, by a law of determinants, is equal to

$$
(-1)^3 |\lambda_1\mu_2\nu_3| . \left\{ \begin{array}{l}
\quad \alpha_1\beta_2\gamma_3 - \alpha_2\beta_1\gamma_3 + \alpha_3\beta_1\gamma_2 \\
- \alpha_1\beta_3\gamma_2 + \alpha_2\beta_3\gamma_1 - \alpha_3\beta_2\gamma_1
\end{array} \right\},
$$

and therefore equal to

$$
(-1)^3 |\lambda_1\mu_2\nu_3| . |\alpha_1\beta_2\gamma_3| .
$$

7. Spottiswoode next considers the case where the two determinants to be multiplied are identical, and where there now comes into play the second half of the law of alternating numbers, namely, that the square of any such number is zero.

He of course readily sees that in this case the permanent on the right is skew symmetric; and from the examples

$$|\lambda_1\mu_2\nu_3|^2 = -\begin{vmatrix} . & \Sigma\lambda\mu & \Sigma\lambda\nu \\ -\Sigma\lambda\mu & . & \Sigma\mu\nu \\ -\Sigma\lambda\nu & -\Sigma\mu\nu & . \end{vmatrix} = 0,$$

$$|\lambda_1\mu_2\nu_3\rho_4\sigma_5|^2 = 0,$$

he concludes that *if the elements of an odd-ordered determinant be alternating numbers, the square of the determinants is zero.* As $\Sigma\lambda\mu$, $\Sigma\lambda\nu$, $\Sigma\mu\nu$, ... are not alternating numbers, this would seem to be equivalent to saying that *a skew symmetric permanent of odd order is zero.* The danger, however, of such hasty deductions and of excessive trust in analogy is here again apparent, for he goes on to formulate the result

$$-|\lambda_1\mu_2\nu_3\rho_4|^2 = \{\Sigma\mu\nu\,.\,\Sigma\lambda\rho + \Sigma\nu\lambda\,.\,\Sigma\mu\rho + \Sigma\lambda\mu\,.\,\Sigma\nu\rho\}^2,$$

where, apart from the above-noted oversight as regards sign, there is a fault in the calculation. The process should, I think, stand as follows :—

$$|\lambda_1\mu_2\nu_3\rho_4|^2 = {}^{+}\begin{vmatrix} . & \Sigma\lambda\mu & \Sigma\lambda\nu & \Sigma\lambda\rho \\ -\Sigma\lambda\mu & . & \Sigma\mu\nu & \Sigma\mu\rho \\ -\Sigma\lambda\nu & -\Sigma\mu\nu & . & \Sigma\nu\rho \\ -\Sigma\lambda\rho & -\Sigma\mu\rho & -\Sigma\nu\rho & . \end{vmatrix}^{+}$$

$$\begin{aligned} &= \Sigma\lambda\mu\,(\Sigma\mu\rho\,.\,\Sigma\lambda\nu - \Sigma\mu\nu\,.\,\Sigma\lambda\rho + \Sigma\lambda\mu\,.\,\Sigma\nu\rho)\,\Sigma\nu\rho \\ &\quad + \Sigma\lambda\nu\,(\Sigma\nu\rho\,.\,\Sigma\lambda\mu + \Sigma\mu\nu\,.\,\Sigma\lambda\rho + \Sigma\lambda\nu\,.\,\Sigma\mu\rho)\,\Sigma\mu\rho \\ &\quad + \Sigma\lambda\rho\,(-\Sigma\lambda\mu\,.\,\Sigma\nu\rho + \Sigma\lambda\nu\,.\,\Sigma\mu\rho + \Sigma\mu\nu\,.\,\Sigma\lambda\rho)\,\Sigma\mu\nu \\ &= (\Sigma\lambda\mu)^2(\Sigma\nu\rho)^2 + (\Sigma\lambda\nu)^2(\Sigma\mu\rho)^2 + (\Sigma\lambda\rho)^2(\Sigma\mu\nu)^2 \\ &\quad + 2\Sigma\lambda\mu\,.\,\Sigma\nu\rho\,.\,\Sigma\lambda\nu\,.\,\Sigma\mu\rho - 2\Sigma\lambda\mu\,.\,\Sigma\nu\rho\,.\,\Sigma\lambda\rho\,.\,\Sigma\mu\nu \\ &\quad + 2\Sigma\lambda\nu\,.\,\Sigma\mu\rho\,.\,\Sigma\lambda\rho\,.\,\Sigma\mu\nu, \end{aligned}$$

where the occurrence of the minus sign precludes the possibility of the right-hand member being expressible as a square.

Capetown, S.A.,
14th March, 1915.

AN INEQUALITY ASSOCIATED WITH THE GAMMA FUNCTION.

By *G. N. Watson.*

THE Weirstrassian definition of $\Gamma(z)$, valid for all values of z (other than negative integers), is effectively equivalent to the formula of Gauss

$$\Gamma(z) = \lim_{n \to \infty} \frac{1 . 2 \ldots n}{z(z+1)\ldots(z+n)} . n^z .$$

The difficulty which arises at the outset of the theory of the Gamma function is the reconciliation of this result with Euler's integral definition

$$\Gamma(z) = \int_0^\infty e^{-t} t^{z-1} \, dt,$$

valid when the *real part of z is positive.*

Write $z = x + iy$; then, when $x > 0$, it is easy to see that the integral

$$\Pi(z, n) = \int_0^n \left(1 - \frac{t}{n}\right)^n t^{z-1} \, dt$$

converges, and, by integrating by parts n times (n being a positive integer), it is readily proved that

$$\Pi(z, n) = \frac{1 . 2 \ldots n}{z(z+1)\ldots(z+n)} n^z .$$

To establish the equivalence of the Gaussian product and the Eulerian integral, it is therefore sufficient to shew that

$$\lim_{n \to \infty} \int_0^n \left(1 - \frac{t}{n}\right)^n t^{z-1} \, dt = \int_0^\infty e^{-t} t^{z-1} \, dt,$$

i.e. that

$$\lim_{n \to \infty} \left[\int_0^n \left\{ e^{-t} - \left(1 - \frac{t}{n}\right)^n \right\} t^{z-1} \, dt + \int_n^\infty e^{-t} t^{z-1} \, dt \right] = 0.$$

This result is proved by Schlömilch (*Höheren Analysis*, p. 243) by some rather elaborate analysis. Bromwich (*Infinite Series*, p. 459) has a simpler proof when $0 < x < 1$; his proof is that

$\int_n^\infty e^{-t} t^{z-1} dt \to 0$ as $n \to \infty$ in virtue of the convergence of $\int_n^\infty e^{-t} t^{z-1} dt$; and he shews by means of the inequality*

$$0 \leq e^{-t} - \left(1 - \frac{t}{n}\right)^n \leq \frac{t}{2n}$$

that the modulus of the first integral tends to zero when $0 < x < 1$, and infers the result for other values of x in virtue of the recurrence formula $\Gamma(z+1) = z\,\Gamma(z)$ satisfied both by the integral and by the limit of the product. Further considerations are necessary when x is an integer and $y \neq 0$.

It is, however, possible to obtain a much more powerful inequality than that due to Bromwich, by quite elementary methods; this inequality is

$$0 \leq e^{-t} - \left(1 - \frac{t}{n}\right)^n \leq e^{-t} \frac{t^2}{n}$$

when $0 \leq t \leq n$ and n is a positive integer; and *this inequality is sufficient to prove that*

$$\lim_{n \to \infty} \int_0^n \left\{e^{-t} - \left(1 - \frac{t}{n}\right)^n\right\} t^{z-1} dt = 0$$

for all values of z such that $x > 0$; for, assuming the inequality, the modulus of the last integral

$$\leq \int_0^n e^{-t} t^2 n^{-1} t^{x-1} dt < n^{-1} \int_0^\infty e^{-t} t^{x+1} dt \to 0$$

since the last integral converges and is independent of n.

To obtain the inequality we proceed thus:

It is obvious that when† $0 \leq v < 1$,

$$1 + v \leq 1 + v + \frac{v^2}{2!} + \frac{v^3}{3!} + \ldots \leq 1 + v + v^2 + v^3 + \ldots$$
$$= 1/(1 - v),$$

writing $v = t/n$, we see that, when $0 \leq t < n$,

$$\left(1 + \frac{t}{n}\right) \leq e^{t/n} \leq \left(1 - \frac{t}{n}\right)^{-1},$$

---

* The inequality is established by Bromwich by an ingenious device based on the consideration of the integral

$$\int_0^t \frac{v}{n}\left(1 - \frac{v}{n}\right)^{n-1} e^v\, dv = \left[-e^v\left(1 - \frac{v}{n}\right)^n\right]_0^t.$$

† The first portion of each of the following inequalities is true when $v = 1$.

so that $$\left(1+\frac{t}{n}\right)^n \leq e^t \leq \left(1-\frac{t}{n}\right)^{-n}.$$

Hence, when $0 \leq t \leq n$, we have

$$\left(1+\frac{t}{n}\right)^{-n} \geq e^{-t} \geq \left(1-\frac{t}{n}\right)^n.$$

Therefore, when $0 \leq t \leq n$,

$$0 \leq e^{-t} - \left(1-\frac{t}{n}\right)^n,$$

$$= e^{-t}\left\{1 - e^t\left(1-\frac{t}{n}\right)^n\right\},$$

$$\leq e^{-t}\left\{1 - \left(1+\frac{t}{n}\right)^n\left(1-\frac{t}{n}\right)^n\right\},$$

$$= e^{-t}\left\{1 - \left(1-\frac{t^2}{n^2}\right)^n\right\}.$$

Now if $1 \geq \alpha \geq 0$, we have $(1-\alpha)^n \geq 1 - n\alpha$, by induction when $1 - n\alpha$ is positive* and obviously when $1 - n\alpha$ is negative. Therefore writing t^2/n^2 for α we get

$$\left(1-\frac{t^2}{n^2}\right)^n \geq 1 - \frac{t^2}{n},$$

and so from the preceding series of inequalities we get

$$0 \leq e^{-t} - \left(1-\frac{t}{n}\right)^n \leq e^{-t}\frac{t^2}{n},$$

which is the required result.

The result is still true if $n \geq 1$, when n is *not* restricted to be an integer, provided that $0 \leq t \leq n$; for the only place in which it was assumed that n was an integer was in proving the inequality $(1-\alpha)^n \geq 1 - n\alpha$, and this is easily proved when $n \geq 1$ and $0 \leq \alpha \leq 1$; for, by Taylor's theorem,

$$(1-\alpha)^n = 1 - n\alpha + \tfrac{1}{2}n(n-1)\alpha^2(1-\theta\alpha)^{n-2},$$

where $0 \leq \theta \leq 1$, and if $n > 1$ the last of the three terms on the right is positive.

* For if $(1-\alpha)^n \geq 1 - n\alpha$, then
$$(1-\alpha)^{n+1} \geq (1-\alpha)(1-n\alpha) = 1 - (n+1)\alpha + n\alpha^2 \geq 1 - (n+1)\alpha.$$

NOTES ON A DIFFERENTIAL EQUATION.

By *G. W. Walker, M.A., F.R.S.*

The differential equation which occurs in the problem of the two dimensional distribution of a gas under the influence of its own gravitation may be written in the form

$$\frac{\partial^2 z}{\partial x} + \frac{\partial^2 z}{\partial y^2} = -4e^z,$$

and a solution of this applicable to real cases is

$$e^z = 2\left\{\left(\frac{\partial \phi}{\partial x}\right)^2 + \left(\frac{\partial \phi}{\partial y}\right)^2\right\}(\phi^2 + \psi^2 + 1)^{-2},$$

where ϕ and ψ are any conjugate functions of x, y, so that

$$\phi + i\psi = f(x + iy).$$

An associated linear equation occurs in dealing with small motion of the system, which is of the form

$$\frac{\partial^2 z}{\partial x^2} + \frac{\partial^2 z}{\partial y^2} = -8z\left\{\left(\frac{\partial \phi}{\partial x}\right)^2 + \left(\frac{\partial \phi}{\partial y}\right)^2\right\}(\phi^2 + \psi^2 + 1)^{-2},$$

and which, on using variables ϕ and ψ, transforms to

$$\frac{\partial^2 z}{\partial \phi^2} + \frac{\partial^2 z}{\partial \psi^2} = -8z(\phi^2 + \psi^2 + 1)^{-2}.$$

A particular solution of this is

$$z = A\left\{\frac{\phi^2 + \psi^2 - 1}{\phi^2 + \psi^2 + 1}\right\}.$$

If we take new variables so that

$$\phi^2 + \psi^2 = \varpi^2, \quad \psi/\phi = \tan\chi,$$

the equation takes the form

$$\frac{1}{\varpi}\frac{\partial}{\partial \varpi}\varpi\frac{\partial z}{\partial \varpi} + \frac{1}{\varpi^2}\frac{\partial^2 z}{\partial \chi^2} = -\frac{8z}{(\varpi^2 + 1)^2}.$$

Hence we have solutions of the form

$$z = {}_{\iota}\Sigma f_n \begin{matrix}\cos\\ \sin\end{matrix} n\chi,$$

where f_n satisfies

$$\frac{1}{\varpi}\frac{\partial}{\partial\varpi}\varpi\frac{\partial f_n}{\partial\varpi} = \left\{\frac{n^2}{\varpi^2} - \frac{8}{(\varpi^2+1)^2}\right\} f_n.$$

Let $\varpi = e^{\zeta}$, then

$$\frac{\partial^2 f_n}{\partial\zeta^2} = (n^2 - 2\,\mathrm{sech}^2\zeta) f_n,$$

so that solutions are

$$f_n = \mathrm{sech}^n\zeta\left(\frac{\partial}{\partial\tanh\zeta}\right)^n \tanh\zeta$$

and
$$f_n = \mathrm{sech}^n\zeta\left(\frac{\partial}{\partial\tanh\zeta}\right)^n (\zeta\tanh\zeta - 1).$$

These may be recognised as associated Toroidal functions of the first and second type.

We may note that for $n \geqq 2$ the solution of the first type fails, but the second form gives a solution for all integral positive values of n.

The complete solution in terms of two arbitrary functions may be found thus:

Writing $\phi + i\psi = \xi$ and $\phi - i\psi = \eta$,

the equation takes the form

$$\frac{\partial^2 z}{\partial\xi\partial\eta} + \frac{2z}{(\xi\eta+1)^2} = 0.$$

By Laplace's method the solution of this is

$$z = \frac{(\xi\eta - 1)}{(\xi\eta + 1)}\{F_1(\xi) + F_2(\eta)\} - \tfrac{1}{2}\{\xi F_1'(\xi) + \eta F_2'(\eta)\},$$

where F_1 and F_2 are arbitrary functions of ξ only and η only, respectively, and F_1' and F_2' are their first derivatives.

A SET OF CRITERIA FOR EXACT DERIVATIVES.

By *E. B. Elliott.*

1. Some time ago* I called attention to a criterion, different from Euler's, which decides whether a rational integral function of a dependent variable and its successive derivatives is or is not the result of differentiating some other. The following is a more complete investigation of such criteria.

Let x be an independent, and $y, z, \ldots$ any number of dependent variables, and let $y_r, z_r, \ldots$ denote $\frac{d^r y}{dx^r}, \frac{d^r z}{dx^r}, \ldots$. We consider, not all functions

$$F(x;\ y, y_1, y_2, \ldots;\ z, z_1, z_2, \ldots;\ \ldots),$$

but an extensive class of such functions, and enquire when an F is a $D^r\phi$, where r is any number, and D is the operator of total differentiation

$$\frac{\partial}{\partial x} + y_1\frac{\partial}{\partial y} + y_2\frac{\partial}{\partial y_1} + \ldots + z_1\frac{\partial}{\partial z} + z_2\frac{\partial}{\partial z_1} + \ldots + \ldots.$$

The advantages of the method are (1) that of applying as simply to cases of many dependent variables as to the case of one, (2) that of providing as definite a single condition for F to be a $D^r\phi$ as for it to be merely a $D\phi$, (3) that of exhibiting as a result of direct operation the associated ϕ when the appropriate condition is satisfied.

The wide class of functions F to which it applies includes all those which can be arranged as sums of parts that are homogeneous, and of degree not zero, in some one set $y, y_1, y_2, \ldots$, and are moreover rational and integral in the derivatives $y_1, y_2, \ldots$ of the set.

If such a sum is a derivative (first or r^{th}) its separate homogeneous parts are so separately, for operation with D or D^r does not alter degree in $y, y_1, y_2, \ldots$, and conversely. The parts may be taken separately.

Accordingly we confine attention to a function

$$u \equiv F(x;\ y, y_1, y_2, \ldots;\ z, z_1, z_2, \ldots;\ \ldots;\ \ldots),$$

which is

(i) homogeneous of degree $i\ (\neq 0)$ in $y, y_1, y_2, y_3, \ldots$;

(ii) rational and integral in $y_1, y_2, y_3, \ldots$.

* "Note on a class of exact differential expressions," *Messenger of Mathematics*, vol. xxv., p. 173.

The limitation (ii) is imposed in order to secure the annihilation of u by some power of the operator ω about to be introduced. Other functions u which are so annihilated are really treated at the same time.

2. Mean by w the greatest sum of y-suffixes in any term of u. As well as the operator D, which is *total*, we use the operator

$$\omega \equiv y\frac{\partial}{\partial y_1} + 2y_1\frac{\partial}{\partial y_2} + 3y_2\frac{\partial}{\partial y_3} + \ldots,$$

which is not total but, like i and w, refers to the set $y, y_1, y_2, \ldots$ only. If there is choice among sets we naturally choose for $y, y_1, y_2, \ldots$ the set of smallest w.

Repeated use will be made of the alternant

$$\omega D - D\omega = y\frac{\partial}{\partial y} + y_1\frac{\partial}{\partial y_1} + y_2\frac{\partial}{\partial y_2} + \ldots,$$

the effect of which on u is to multiply it by its degree in $y, y_1, y_2, \ldots$, however u may also involve x and other sets $z, z_1, z_2, \ldots$, &c.

The criteria to be obtained depend on compound operators of the type

$$(r, n) \equiv (D\omega - ri)(D\omega - r+1\,.\,i)\ldots(D\omega - ni),$$

where r, n are not fractional, and $0 \leqq r \leqq n \leqq w$. For instance (r, r) is $D\omega - ri$, and $(0, n)$ is $D\omega\,(D\omega - i)\ldots(D\omega - ni)$.

The main facts are as follows:—

LEMMA. *For any* n, $(0, n) = D^{n+1}\omega^{n+1}$.

THEOREM (0). $(0, w)\,u$ *vanishes identically.*

THEOREM (r), for $r = 1, 2, \ldots, w$. *According as* $(r, w)\,u$ *is or is not identically zero,* u *is or is not an* r^{th} *derivative* $D^r v$ *at least.*

THEOREM $(w+1)$. u *cannot be a* $(w+1)^{th}$ *derivative.*

The last of these theorems is at once clear. In fact, if $u' = Dv'$ is integral in $y_1, y_2, y_3, \ldots, v'$ cannot be fractional in them. Also the greatest sum of y-suffixes in u' exceeds by 1 the greatest in v'. Thus in a $D^{w+1}v$, where v involves any of $y, y_1, y_2, \ldots$, the greatest sum of y-suffixes in a term cannot be less than $w+1$, whereas in our u the greatest sum is w.

Another immediate fact is the half of theorem (1) that u is a Dv when $(1, w)\,u = 0$. For upon transposition this relation expresses $w!\,.\,i^w u$ as a $D\omega\,\{\ldots\}$.

3. We will now prove the *lemma*, and deduce *theorem* (0).

The lemma is true for $n=1$. For an operation with ω or D does not alter y-degree i, so that use of the alternant above gives

$$(0,\,1)\,u \equiv D\omega\,(D\omega - i)\,u = D\,\{\omega D\,.\,\omega - i\omega\}\,u$$
$$= D\,\{(D\omega + i)\,\omega - i\omega\}\,u = D^2\omega^2 u.$$

We have then only to prove that, if true for $n-1$, it is for n. Now, on the assumption foı $n-1$ we have

$$(0,\,n) = D^n\omega^n\,(D\omega - ni) = D^n\omega^{n-1}\,\{(D\omega + i)\,\omega - ni\omega\}$$
$$= D^n\omega^{n-1}\,\{D\omega - \overline{n-1}\,.\,i\}\,\omega$$
$$= D^n\omega^{n-2}\,\{D\omega - \overline{n-2}\,.\,i\}\,\omega^2$$
$$= \dots\dots\dots\dots\dots\dots\dots\dots$$
$$= D^n\,.\,D\omega\,.\,\omega^n = D^{n+1}\omega^{n+1}.$$

The deduction of theorem (0) is immediate. u contains no term with sum of y-suffixes exceeding w. Now operation with ω lowers sum of y-suffixes by 1, and annihilates terms free from $y_1, y_2, \dots$. Thus $\omega^{w+1}u=0$, and so $(0,\,w)\,u$, *i.e.* $D^{w+1}\omega^{w+1}u,=0$.

4. We next prove the half of the general theorem (r), that if u is a $D^r v$, then $(r,\,w)\,u=0$.

It has already been seen that if w is the greatest sum of y-suffixes in any term of $u\equiv D^r v$, then the greatest sum in any term of v must be $w-r$. Thus $\omega^{w-r+1}v=0$, and so $(0,\,w-r)\,v=0$.

Now by repeated use, moving backwards, of

$$(D\omega - ni)\,D = D\,(D\omega + i) - niD = D\,(D\omega - \overline{n-1}\,.\,i)$$

we obtain, with $u=D^r v$,

$$(r,\,w)\,u \equiv (D\omega - ri)\,(D\omega - \overline{r+1}\,.\,i)\,\dots\,(D\omega - wi)\,D^r v$$
$$= D(D\omega = \overline{r-1}\,.\,i)(D\omega - ri)\dots(D\omega - \overline{w-1}\,.\,i)\,D^{r-1}v$$
$$= D\,(r-1,\,w-1)\,D^{r-1}v$$
$$= D^2(r-2,\,w-2)\,D^{r-2}v$$
$$= \dots\dots\dots\dots\dots\dots\dots\dots$$
$$= D^r\,(0,\,w-r)\,v = 0.$$

5. Conversely, we have to prove that if $(r,\,w)\,u=0$, then u is a $D^r v$. This follows from a fact, to be proved in the

next article, that any u of given i and w can always be expressed as a sum of $w+1$ parts, some perhaps vanishing,

$$u = \{A_0(1, w) + A_1 D\omega(2, w) + \ldots \\ + A_{w-1} D^{w-1}\omega^{w-1}(w, w) + A_w D^w \omega^w\} u,$$

where the coefficients are definite numerical constants, and the $(r+1)^{\text{th}}$ part, for $r = 1, 2, \ldots, w$, is expressed as an r^{th} derivative.

The deduction from the fact is as follows. Given $(r, w)u = 0$, we have also

$$(r-1, w)u = (D\omega - r - 1 . i)(r, w)u = 0,$$

and in succession

$$(r-2, w)u = 0, \quad (r-3, w)u = 0, \quad \ldots (1, w)u = 0.$$

Thus $(r, w)u = 0$ necessitates

$$u = D^r \{A_r \omega^r (r+1, w) + A_{r+1} D\omega^{r+1}(r+2, w) + \ldots + A_w D^{w-r}\omega^w\} u.$$

6. The precise theorem of separation of which a part has been used may be thus stated:

THEOREM. *Any u such as specified at the end of* § 1, *and in fact any u of non-zero y-degree i throughout which satisfies $\omega^{w+1}u = 0$ for some w, can be expressed by direct operation as a sum of $w+1$ parts (some perhaps zero) each satisfying one of the equations $(D\omega - ri)u = 0$, for $r = 0, 1, 2, \ldots, w$; and of these $w+1$ parts, the first, u_0, is not a derivative, while generally the $(r+1)^{th}$, u_r, is an r^{th} but not an $(r+1)^{th}$ derivative.*

It is not stated (or true) that in all cases the parts $u_0, u_1, \ldots, u_w$ involve no higher derivatives than the sum u does. The separation, like others which I have considered elsewhere, is effected by use of the identity among polynomials

$$1 = \sum_{s=0}^{s=n} \left\{\frac{1}{F'(a_1)} \cdot \frac{1}{z-a}\right\} F(z),$$

where $\qquad F(z) \equiv (z-a_0)(z-a_1)\ldots(z-a_n).$

Taking w for n, particular values for $a_0, a_1, \ldots a_n$, and the operator $D\omega$ for z, this tells us that

$$u = \left\{\frac{A_0}{D\omega} + \frac{A_1}{D\omega - i} + \frac{A_2}{D\omega - 2i} + \ldots + \frac{A_w}{D\omega - wi}\right\} \\ \times D\omega(D\omega - i)\ldots(D\omega - wi)u,$$

where, for $r = 0, 1, 2, \ldots w$,

$$A_r = (-1)^{w-r} i^{-w} \frac{1}{r!(w-r)!}.$$

u is thus written as a sum of $w+1$ parts, of which the $(r+1)^{\text{th}}$, for each r, is presented in a form which shows that it satisfies

$$(D\omega - ri)\, u_r = A_r(0,\, w)\, u = 0.$$

Also, for each r,

$$u_r = A_r D\omega\,(D\omega - i)\ldots(D\omega - \overline{r-1}\,.\,i).(D\omega - \overline{r+1}\,.\,i)\ldots(D\omega - wi)\, u$$
$$= A_r D^r\omega^r\,(r+1,\, w)\, u,$$

where the final $(w+1,\, w)$ denotes 1.

Thus so much of the theorem as was required in § 5 has been proved.

7. The theorem, however, states further that the first part u_0, when it does not vanish, is not a derivative, and that the $(r+1)^{\text{th}}$ part u_r, presented above as an r^{th} derivative, is not an $(r+1)^{\text{th}}$.

Now u_r satisfies $(D\omega - ri)\, u_r = 0$. If it were an $(r+1)^{\text{th}}$ derivative it would also, by theorem $(r+1)$, have to satisfy

$$(D\omega - \overline{r+1}\,.\,i)\,(D\omega - \overline{r+2}\,.\,i)\ldots(D\omega - wi)\, u_r = 0.$$

But this equation is inconsistent with the other. For on substituting in it, from the other, $ri\, u_r$ for $D\omega\, u_r$ we arrive at the unsatisfied

$$(-i)\,(-2i)\ldots(\overline{r-w}\,.\,i)\, u_r = 0.$$

This applies for $r = 0, 1, 2, \ldots w-1$, but not for $r = w$. However u_w, *i.e.* $A_w D^w\omega^w u$, has the same w as u, and cannot be a $(w+1)^{\text{th}}$ derivative by theorem $(w+1)$.

Corollary 1. Any u satisfying $(D\omega - ri)\, u = 0$ is an r^{th} derivative, by theorem (r), and not an $(r+1)^{\text{th}}$ by the above. It must *not* be assumed conversely that every r^{th}, but not $(r+1)^{\text{th}}$, derivative satisfies this equation.

Corollary 2. $(D\omega - k)\, u = 0$ cannot be satisfied except for one of the values $0, i, 2i, \ldots wi$ of k.

Corollary 3. If $\omega u = 0$ (with $i \neq 0$) u cannot be a derivative. For if it were $(1,\, w)\, u = 0$ would give $w!\, i^w u = 0$. [N.B. This does not apply with $i = 0$. If $i = 0$ for v we have $(\omega D - D\omega)\, v = 0$, and so $\omega Dv = 0$ whenever $\omega v = 0$ or c. For instance ω annihilates all of

$$D^n(y_1/y),\ \textit{i.e.}\ D^{n+1}\log y,\ \ D^{n+1}\log(yy_2 - y_1^{\,2}),$$
$$D^n\phi\,(x;\ z, z_1, z_2, \ldots;\ D^3\log y).]$$

Corollary 4. $(1,\, w)\, u$, when not zero, is annihilated by $D\omega$, and so, with $i \neq 0$, by ω; and generally $(r,\, w)\, u$, with $i \neq 0$, is annihilated by ω^r.

8. The process for examining a given u may proceed as follows:—

Form $D\omega u$. If this is a multiple of u, then u is exhibited as a Dv. In such a case the multiplier must be one of i, $2i$, ... wi. If $D\omega u = riu$, then u is an r^{th} but not an $(r+1)^{\text{th}}$ derivative. In particular if $D\omega u = wiu$, then u is a w^{th} derivative. [*Example.* $y^{-\frac{11}{3}}$ is an integrating factor of $9y^2y_3 - 45yy_1y_2 + 40y_1{}^3 = 0$, which makes it an exact $D^3v = 0$.]

If no $(D\omega - ri)\,u = 0$, take $(D\omega - wi)\,u \equiv u'$, and operate on it with $D\omega$. If the result is a multiple of u', the multiple must be ri, with r one of $1, 2, \ldots w-1$ {not w because $(0, w-1)\,u' = 0$ and $(wi-i)(wi-2i)\ldots(wi - \overline{w-1}\,.\,i)\,u \neq 0$}. If it be ri, then u is an r^{th} derivative by theorem (r).

If, however, no $(D\omega - ri)(D\omega - wi)\,u = 0$, take $(D\omega - \overline{w-1}\,.\,i)(D\omega - wi)\,u \equiv u''$, and form $D\omega u''$. If this is an riu'', with r necessarily one of $1, 2, \ldots w-2$, then u is an r^{th} derivative.

If not, proceed again in like manner. Finally, if $(D\omega - 2i)(D\omega - 3i)\ldots(D\omega - wi)\,u \neq 0$, then u is a first derivative (only) or not a derivative according as this is or is not annihilated by $D\omega - i$.

9. A few words as to the excluded case of $i = 0$ may be added.

The substitution of $e^{y'} = y$ in this case gives to

$$u \equiv F(x;\; y, y_1, y_2, \ldots;\; z, z_1, z_2, \ldots;\; \ldots)$$

the form free from y'

$$u \equiv f(x;\; y_1', y_2', \ldots;\; z, z_1, z_2, \ldots;\; \ldots),$$

where notice that f would also be free from y_1' if $\omega F = 0$ were satisfied.

It is not to be expected that f will be homogeneous in $y_1', y_2', \ldots$. But if it is a derivative (first or r^{th}) its various homogeneous parts must be so separately, and conversely. Deal with them one at a time.

If, according to § 1, we are still concerned with functions F which are rational and integral in $y_1, y_2, \ldots$, so that their homogeneity of degree 0 arises from negative powers of y as factors of terms, the functions f will be rational and integral in $y_1', y_2', \ldots$, and they do not involve y'. Any part of f which may be of degree zero must be free from all of $y', y_1', y_2', \ldots$. If it be a constant or a $\phi(x)$, it is of course a derivative of any order. If it be a $\phi(x;\; z, z_1, z_2, \ldots;\; \ldots)$, it must be examined by consideration of another set $z, z_1, z_2, \ldots$

Our conclusions, however, with regard to functions F with $i \neq 0$ have applied, not only to functions rational and integral

in $y_1, y_2, \ldots$, but to other functions annihilated by some power ω^{w+1} of ω. The functions f which are the transformations of these are annihilated by $\left(\frac{\partial}{\partial y_1'}\right)^{w+1}$, and so are rational and integral in y_1' though not in the whole set $y_1', y_2', \ldots$. A part of f of degree 0 in the set may now involve $y_1', y_2', \ldots$. and we may proceed by use of a second exponential transformation applied to y_1'.

To the homogeneous parts of f with non-zero degrees criteria such as have been developed apply, using $y', y_1', y_2', \ldots$ instead of $y, y_1, y_2, \ldots$. The applicability is complete in the ordinary cases of f rational and integral in $y_1', y_2', \ldots$; but in the additional cases of f rational and integral in y_1', while not so in the set, there is applicability only when f, or a part of it in question, is annihilated by some power of

$$\omega' \equiv y' \frac{\partial}{\partial y_1'} + 2y_1' \frac{\partial}{\partial y_1'^2} + \ldots.$$

A TRANSFORMATION OF CENTRAL MOTION.

By *R. Hargreaves, M.A.*

THE title is used because the suggestion is derived from the problem of central motion and the examples given are connected with it, but the transformation is not limited to that problem.

§ 1. If the expression $\frac{1}{2}(\dot{r}^2 + r^2\dot{\theta}^2 + r^2\dot{\phi}^2 \sin^2\theta)$ is used for kinetic energy, and no preliminary argument is adduced to shew that the motion must be plane, then the kinetic potential on ignoring ϕ is

$$L = \tfrac{1}{2}(\dot{r}^2 + r^2\dot{\theta}^2) - \frac{\kappa^2}{2r^2\sin^2\theta} - F(r) \;\ldots\ldots\ldots\ldots(1).$$

But this may be regarded as defining a problem of plane motion with θ an angle in the plane, a problem in which the central force is supplemented by a repulsive force κ^2/y^3 in a fixed direction y. We have then the assurance that this problem can be made to depend on that of

$$L = \tfrac{1}{2}(\dot{r}^2 + r^2\dot{\theta}'^2) - F(r)$$

by a transformation, which in fact is

$$\cos\theta' \cos\alpha = \cos\theta,\ \sin\theta' = \sin\theta \sin\phi,\ \cos\theta' \sin\alpha = \sin\theta\cos\phi \ldots(2).$$

Here θ is a polar distance measured from OZ, ϕ an azimuth

measured from a plane ZOX perpendicular to the plane of motion, θ' an angle in the plane of motion measured from its intersection with ZOX, and α the angle between OZ and this line of intersection.

The new orbit is derived from the original by attributing to any radius an angle in the plane, which in the primary orbit stood as polar distance for the same radius.

The general position for central forces is that the kinetic potential

$$\left.\begin{aligned} &L = \tfrac{1}{2}(\dot{r}^2 + r^2\dot{\theta}^2) - F(r) - \frac{f(\theta)}{2r^2} \\ &\text{makes } r \text{ depend on } L_0 = \tfrac{1}{2}\dot{r}^2 - F(r) - \frac{k^2}{2r^2}; \\ &\text{while } h \text{ being } r^2\dot{\theta},\quad h^2 + f(\theta) = k^2 \text{ (constant)}, \\ &\text{and } E_0 \text{ (constant)} = \tfrac{1}{2}\left(\dot{r}^2 + \frac{k^2}{r^2}\right) + F(r) \\ &\qquad = \tfrac{1}{2}\left[k^2u^2 + \{k^2 - f(\theta)\}\left(\frac{du}{d\theta}\right)^2\right] + F(r) \end{aligned}\right\}\text{..(3)},$$

where $ru = 1$. Also the connexion with the problem

$$\left.\begin{aligned} &L' = \tfrac{1}{2}(\dot{r}^2 + r^2\dot{\theta}'^2) - F(r) \\ &\text{is given by } h\,d\theta' = k\,d\theta, \text{ or } d\theta' = k\,d\theta/\sqrt{\{k^2 - f(\theta)\}} \end{aligned}\right\}\text{...(4)}.$$

A more general proposition is that if T_0 and U_0 be kinetic and potential energies for coordinates $x_1 \dots x_n$, X a positive function of these co-ordinates, then

$$\left.\begin{aligned} &L = T_0 - U_0 + \frac{X}{2}\dot{\theta}^2 - \frac{f(\theta)}{2X} \\ &\text{makes } x_1 \dots x_n \text{ depend on } \quad L_0 = T_0 - U_0 - \frac{k^2}{2X}; \\ &\text{while} \qquad (X\dot{\theta})^2 + f(\theta) = k^2, \\ &\text{and} \qquad E_0 \text{ (constant)} = T_0 + U_0 + \frac{k^2}{2X}. \end{aligned}\right\}\text{......(5)},$$

$$\left.\begin{aligned} &\text{The connexion with the problem } L' = T_0 + U_0 + \frac{X}{2}\dot{\theta}'^2 \\ &\text{is given by } X\dot{\theta}' = k, \text{ making } d\theta' = k\,d\theta/\sqrt{\{k^2 - f(\theta)\}} \end{aligned}\right\}\text{... (6)},$$

exactly as in (4). The θ component of force in (5) gives $\dfrac{d}{dt}(X\dot{\theta}) + \dfrac{f'(\theta)}{2X} = 0$, the integral of which is used in (5).

The x_1 component of force has a term

$$\frac{\partial}{\partial x_1}\left(\frac{X}{2}\dot{\theta}^2 - \frac{f(\theta)}{X}\right) = \tfrac{1}{2}\left(\dot{\theta}^2 + \frac{f(\theta)}{X^2}\right)\frac{\partial X}{\partial x_1} = -\frac{\partial}{\partial x_1}\frac{k^2}{2X}$$

in agreement with derivation from L_0. If $f(\theta)$ is negative we may have $-k^2$ for k^2. The most general form reducible to (5) is $L = T_0 - U_0 + \frac{X}{2}a\dot{\theta}_1^2 + XT_2$, where a and the coefficients of a quadric T_2 in $\dot{\theta}_2 \dots \dot{\theta}_m$ may be dependent on θ_1, but not on $\theta_2 \dots \theta_m$.

§ 2. For the problem (1) with which we started and $F(r) = -\mu/r$, the orbit is

$$\left.\begin{aligned} l/r &= 1 + \{B\cos\theta \pm A\sqrt{(\sin^2\theta - \sin^2\alpha)}\}/\cos\alpha \\ \text{with}\quad k^2 &= \mu l,\ \text{and}\ 2E_0 l^2/k^2 = A^2 + B^2 - 1, \\ f(\theta) &= k^2\sin^2\alpha/\sin^2\theta,\quad h\sin\theta = k\sqrt{(\sin^2\theta - \sin^2\alpha)} \end{aligned}\right\}\dots(7).$$

The case $B = 0$ is symmetrical with respect to the line $\theta = \frac{\pi}{2}$; the case $A = 0$ represents part of an ellipse or hyperbola traversed from $\theta = \alpha$ to $\theta = \pi - \alpha$ and back.

The condition for the existence of asymptotes is $A^2 + B^2 > 1$. If $A^2 + B^2 < 1$ then as A increases from 0 the single line becomes a closed curve with $\theta = \alpha$ and $\theta = \pi - \alpha$ for tangents, and two points of inflexion on the $+A$ section so long as $A < \cot^2\alpha$, but for $A > \cot^2\alpha$ the oval is convex.

For $A^2 + B^2 > 1$ there are two asymptotes which for B positive and < 1 both lie on the $-A$ section, the $+A$ section connecting the tangents $\theta = \alpha$ and $\theta = \pi - \alpha$ by a finite arc. For $A^2 + B^2 > 1$, but B positive and > 1, one asymptote lies in the $+A$ section the other in the $-A$ section; in this case the tangent $\theta = \pi - \alpha$ does not belong to the orbit, which only proceeds to the asymptote on the $+A$ section, that asymptote for which θ is nearest $\pi - \alpha$. The change of sign in B corresponds to writing $\pi - \theta$ for θ.

The problem of repulsion from each of two axes at right-angles is solved by taking

$$\left.\begin{aligned} &\frac{f(\theta)}{k^2} = \frac{\cos^2\alpha\cos^2\beta}{\cos^2\theta} + \frac{\sin^2\alpha\sin^2\beta}{\sin^2\theta}, \\ &\text{which corresponds to the transformation} \\ &\sin^2\theta = \sin^2\beta\sin^2\theta' + \sin^2\alpha\cos^2\theta', \\ &\text{and makes } h^2\sin^2\theta\cos^2\theta = k^2(\sin^2\beta - \sin^2\theta)(\sin^2\theta - \sin^2\alpha) \end{aligned}\right\}.(8),$$

where $\beta > \alpha$. The angles α and β may be found in terms of the ratios L/k^2 and M/k^2 when the repulsive forces are L/x^3 and M/y^3; and if $L=M$, $\beta=\frac{\pi}{2}-\alpha$. The independent solution is

$$l/r = 1 + \{\pm A\sqrt{(\sin^2\theta - \sin^2\alpha)} \pm B\sqrt{(\sin^2\beta - \sin^2\theta)}\}/\sqrt{(\sin^2\beta - \sin^2\alpha)} \ldots (9),$$

with $k^2 = \mu l$, and $2E_0 l^2/k^2 = A^2 + B^2 - 1$ as in the last case; and here also the condition for asymptotes is $A^2 + B^2 > 1$.

The typical form when $A^2 + B^2 < 1$ is a distorted figure of eight, and the lines $\theta = \alpha$, $\theta = \beta$ touch the curve, each at two points. In the case with asymptotes one or both of the more distant of these points of contact may be excluded, the exclusion turning on A or B being separately greater than 1. If we follow the curve from each asymptotic end the sections cross and form a loop touching $\theta = a$ and $\theta = \beta$ at the less distant points.

When the primary orbit is an ellipse to the centre, *i.e.* $F(r) = \mu r^2/2$, these transformations give respectively for the orbits

$$\left.\begin{aligned} \cos^2\alpha/r^2 &= A(\sin^2\theta - \sin^2\alpha) + B\cos^2\theta \pm 2C\cos\theta\sqrt{(\sin^2\theta - \sin^2\alpha)} \\ (\sin^2\beta - \sin^2\alpha)/r^2 &= A(\sin^2\theta - \sin^2\alpha) + B(\sin^2\beta - \sin^2\theta) \\ &\qquad \pm 2C\sqrt{\{(\sin^2\theta - \sin^2\alpha)(\sin^2\beta - \sin^2\theta)\}}, \\ \text{with} \quad \mu/k^2 &= AB - C^2, \text{ and } 2E_0/k^2 = A + B \end{aligned}\right\}$$

in each case; forms which present less variety than those derived from focal orbits.

In these examples the centre of force lies outside the new orbit. But if we take $f(\theta) = k^2\sin^2\epsilon\sin^2\theta$, we get

$$\theta' = \int_0^\theta d\theta/\sqrt{(1 - \sin^2\epsilon\sin^2\theta)},$$

and a complete circuit for θ is possible. So also for

$$f(\theta) = -2ma\cos\theta \text{ when } 2ma < k^2,$$

and
$$d\theta' = kd\theta/\sqrt{(k^2 + 2ma\cos\theta)};$$

i.e. we have an exact solution for an attracting centre in combination with a doublet at the origin.

ON THE STEADY MOTION OF FLUID UNDER GRAVITY.

By *W. Burnside.*

THE steady motion which corresponds to Rankine's trochoidal waves is an instance of a two-dimensional rotational motion of a fluid under gravity for which the stream lines are lines of constant pressure.

If x, y are measured horizontally and vertically downwards, and ψ is the stream-function, this motion is determined, with suitable units, by the equations

$$x = \theta + e^{-r}\cos\theta,$$

$$y = r - e^{-r}\sin\theta,$$

$$\psi = \sqrt{g}\,(r + \tfrac{1}{2}e^{-2r}).$$

These equations in fact give

$$\left(\frac{\partial\psi}{\partial x}\right)^2 + \left(\frac{\partial\psi}{\partial y}\right)^2 - 2gy = g\,(1 - 2r + e^{-2r}) = f(\psi).$$

It is not, I believe, known that the only steady two-dimensional irrotational motion of a fluid under gravity for which the stream-lines are lines of constant pressure is a uniform horizontal stream.

Assuming the existence of such a motion, and taking the origin at a point in the fluid which is not a point of zero-velocity, the motion in the neighbourhood of the origin must be given by

$$z = \alpha_1 w + \alpha_2 w^2 + \alpha_3 w^3 + \dots \quad \dots\dots\dots\dots\dots \text{(i)},$$

where $\qquad z = x + iy,\quad w = \phi + i\psi,\quad \alpha_1 \neq 0,$

and the series on the right-hand side of (i) is absolutely convergent so long as $|w|$ does not exceed some finite positive quantity. If the axes are taken horizontally and vertically downwards the pressure equation is

$$\frac{p + C}{\rho} = gy - \tfrac{1}{2}q^2 \dots\dots\dots\dots\dots\dots\dots \text{(ii)},$$

and if the pressure is constant along each stream-line, the pressure in the neighbourhood of the origin must be given by

$$\frac{p + C}{g\rho} = p_0 + p_1\psi + p_2\psi^2 + \dots,$$

where the series is absolutely convergent so long as $|\psi|$ does

not exceed some finite positive quantity. If w and $\bar{w}$ are conjugate imaginaries,

$$\frac{1}{q^2} = \frac{dz}{dw}\frac{d\bar{z}}{d\bar{w}} = (\alpha_1 + 2\alpha_2 w + 3\alpha_3 w^2 + \ldots)(\bar{\alpha}_1 + 2\bar{\alpha}_2 \bar{w} + 3\bar{\alpha}_3 \bar{w}^2 + \ldots),$$

$$y = \frac{1}{2i}(z - \bar{z}) = \frac{1}{2i}(\alpha_1 w - \bar{\alpha}_1 \bar{w} + \alpha_2 w^2 - \bar{\alpha}_2 \bar{w}^2 + \ldots),$$

$$\psi = \frac{1}{2i}(w - \bar{w}).$$

Now equation (ii) implies that, for sufficiently small values of $|w|$, the relation

$$(p_0 - y + p_1\psi + p_2\psi^2 + \ldots)\frac{dz}{dw}\frac{d\bar{z}}{d\bar{w}} + \frac{1}{2g} = 0,$$

with the preceding values entered, must be an identity. Writing the earlier terms at length this identity is

$$\Big[p_0 + \frac{p_1 - \alpha_1}{2i} w - \frac{p_1 - \bar{\alpha}_1}{2i}\bar{w} - \left(\frac{p_2}{4} + \frac{\alpha_2}{2i}\right) w^2$$
$$+ \frac{p_2}{2} w\bar{w} - \left(\frac{p_2}{4} - \frac{\bar{\alpha}_2}{2i}\right)\bar{w}^2 - \left(\frac{p_3}{8i} + \frac{\alpha_3}{2i}\right) w^3$$
$$+ \frac{3p_3}{8i} w^2\bar{w} - \frac{3p_3}{8i} w\bar{w}^2 + \left(\frac{p_3}{8i} + \frac{\bar{\alpha}_3}{2i}\right)\bar{w}^3 + \ldots\Big]$$
$$\times\Big[\alpha_1\bar{\alpha}_1 + 2\alpha_2\bar{\alpha}_1 w + 2\bar{\alpha}_2\alpha_1\bar{w} + 3\alpha_3\bar{\alpha}_1 w^2 + 4\alpha_2\bar{\alpha}_2 w\bar{w} + 3\bar{\alpha}_3\alpha_1\bar{w}^2$$
$$+ 4\alpha_4\bar{\alpha}_1 w^3 + 6\alpha_3\bar{\alpha}_2 w^2\bar{w} + 6\bar{\alpha}_3\alpha_2 w\bar{w}^2 + 4\bar{\alpha}_4\alpha_1\bar{w}^3 + \ldots.\Big]$$
$$+ \frac{1}{2g} \equiv 0.$$

Equating to zero the coefficients of w, $\bar{w}$, w^2, $w\bar{w}$, $\bar{w}^2$, and taking into account that $\alpha_1 \neq 0$,

$$\left.\begin{aligned}
p_0\alpha_2 &= -\frac{p_1 - \alpha_1}{4i}\alpha_1\\
p_0\bar{\alpha}_2 &= \frac{p_1 - \bar{\alpha}_1}{4i}\bar{\alpha}_1\\
3p_0\alpha_3 + 2\frac{p_1 - \alpha_1}{2i}\alpha_2 - \left(\frac{p_2}{4} + \frac{\alpha_2}{2i}\right)\alpha_1 &= 0\\
3p_0\bar{\alpha}_3 - 2\frac{p_1 - \bar{\alpha}_1}{2i}\bar{\alpha}_2 - \left(\frac{p_2}{4} - \frac{\bar{\alpha}_2}{2i}\right)\bar{\alpha}_1 &= 0\\
4p_0\alpha_2\bar{\alpha}_2 + \frac{p_1 - \alpha_1}{2i}2\bar{\alpha}_2\alpha_1 - \frac{p_1 - \bar{\alpha}_1}{2i}2\alpha_2\bar{\alpha}_1 + \frac{p_2}{2}\alpha_1\bar{\alpha}_1 &= 0
\end{aligned}\right\}\ldots\text{(iii)}.$$

These relations determine α_2, $\bar{\alpha}_2$, α_3, $\bar{\alpha}_3$, and p_2 in terms of α_1, $\bar{\alpha}_1$, p_0 and p_1. When the coefficients of the terms of the third order in w and $\bar{w}$ are equated to zero, there are four additional equations and only three more coefficients, so that a relation between α_1, $\bar{\alpha}_1$, p_0, and p_1 must arise. The coefficient of $w^2\bar{w}$, omitting terms which obviously cancel, gives the relation

$$\frac{3p_3}{8i}\alpha_1 + p_2\alpha_2 - 3\frac{p_1-\bar{\alpha}_1}{2i}\alpha_3 = 0 \quad\text{..................(iv).}$$

From the preceding relations (iii)

$$p_0p_2 = \tfrac{1}{2}(p_1-\alpha_1)(p_1-\bar{\alpha}_1)$$

and
$$3\frac{p_1-\bar{\alpha}_1}{2i}\alpha_3 = -\frac{p_1-\bar{\alpha}_1}{2ip_0}\left[2\frac{p_1-\alpha_1}{2i}\alpha_2 - \left(\frac{p_2}{4}+\frac{\alpha_2}{2i}\right)\alpha_1\right]$$

$$= \tfrac{1}{2}\frac{(p_1-\alpha_1)(p_1-\bar{\alpha}_1)}{p_0}\alpha_2 + \frac{p_1-\bar{\alpha}_1}{2ip_0}\left(\frac{p_2}{4}+\frac{\alpha_2}{2i}\right)\alpha_1.$$

The above relation (iv) then becomes

$$3p_3 - \frac{p_1-\bar{\alpha}_1}{p_0}\left(p_2+\frac{2\alpha_2}{i}\right) = 0,$$

or, entering their values for p_2 and α_2,

$$6p_0^2p_3 - (p_1-\bar{\alpha}_1)\left[(p_1-\alpha_1)(p_1-\bar{\alpha}_1) + (p_1-\alpha_1)\alpha_1\right] = 0.$$

i.e.
$$6p_0^2p_3 - (p_1-\alpha_1)(p_1-\bar{\alpha}_1)(p_1+\alpha_1-\bar{\alpha}_1) = 0.$$

The constants p_0, p_1, etc., are essentially real, and therefore α_1 must be real.

Hence, at every point in the fluid which is not a point of zero-velocity, the fluid velocity is horizontal. This condition is satisfied only by a uniform horizontal stream.

The following considerations give an independent proof of the same result. If ψ is the stream-function for a steady irrotational motion of a fluid under gravity, in which the stream lines are lines of constant pressure, while the force-potential is a function of y only, then

$$\frac{\partial^2\psi}{\partial x^2} + \frac{\partial^2\psi}{\partial y^2} = 0 \quad\text{......................(v),}$$

$$\left(\frac{\partial\psi}{\partial x}\right)^2 + \left(\frac{\partial\psi}{\partial y}\right)^2 = g(y) + f(\psi)\quad\text{...........(vi),}$$

hold simultaneously. Since ψ satisfies (v), so also does

$$\log\left\{\left(\frac{\partial\psi}{\partial x}\right)^2 + \left(\frac{\partial\psi}{\partial y}\right)^2\right\}.$$

This gives the equation

$$\{(g+f)f''-f'^2\}(g+f)+g''(g+f)-g'^2-2g'f'\frac{d\psi}{dy}=0,$$

where f' and f'' are the first and second differential coefficients of $f(\psi)$ with respect to ψ, and g' and g'' are the first and second differential coefficients of $g(y)$ with respect to y. The last equation is of the form

$$\frac{\partial\psi}{\partial y}=F(\psi, y) \text{ ....}(\text{vii}),$$

Hence, from (vi),

$$\left(\frac{\partial\psi}{\partial x}\right)^2=g+f-\{F(\psi, y)\}^2 \text{ ....}(\text{viii}).$$

Differentiating (vii) and (viii), with respect to y and x respectively,

$$\frac{\partial^2\psi}{\partial y^2}=\frac{\partial F}{\partial y}+\frac{\partial F}{\partial\psi}\frac{\partial\psi}{\partial y},$$

$$\frac{\partial^2\psi}{\partial x^2}=\tfrac{1}{2}f'-F\frac{\partial F}{\partial\psi}.$$

Adding these equations, and taking account of (v) and (vii), there results

$$\tfrac{1}{2}f'+\frac{\partial F}{\partial y}=0,$$

so that ψ is a function of y only. The motion is therefore a uniform stream.

A TRANSFORMATION OF THE PARTIAL DIFFERENTIAL EQUATION OF THE SECOND ORDER.

By *J. R. Wilton, M.A., D.Sc.*

The equation

$$r=f(x, y, z, p, q, s, t) \text{ ....}(1)$$

may, by the properties of equations in involution, be transformed into a partial differential equation of the second order which is linear in the derivatives of the second order. In general this new equation will involve five independent variables x, y, z, p, and q; but in particular cases it may involve only two, which may be either x and y or p and q.

We assume

$$\left.\begin{aligned} t&=\phi(x, y, z, p, q, s, \lambda) \\ \text{and therefore}\quad r&=f(x, y, z, p, q, s, \phi)\end{aligned}\right\}=1 \text{ ....}(2),$$

and we determine ϕ by solving the ordinary equation of the first order

$$\frac{\partial\phi}{\partial s}\left\{\frac{\partial f}{\partial s}+\frac{\partial f}{\partial\phi}\frac{\partial\phi}{\partial s}\right\}=1\ldots\ldots\ldots\ldots\ldots(3),$$

in which x, y, z, p, and q are treated as constants. The solution will involve an arbitrary function of x, y, z, p, and q, which we call λ.

Putting $f=\theta\,(x, y, z, p, q, s, \lambda)$

we have

$$\frac{\partial\theta}{\partial s}\frac{\partial\phi}{\partial s}=1\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(4);$$

and the elimination of λ between the equations

$$r=\theta,\quad t=\phi\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(5)$$

leads to the original equation (1).

Let

$$\left(\frac{d}{dx}\right)\equiv\frac{\partial}{\partial x}+p\frac{\partial}{\partial z}+\theta\frac{\partial}{\partial p}+s\frac{\partial}{\partial q},$$

$$\left(\frac{d}{dy}\right)\equiv\frac{\partial}{\partial y}+q\frac{\partial}{\partial z}+s\frac{\partial}{\partial p}+\phi\frac{\partial}{\partial q}.$$

Then differentiating the first of equations (5) with regard to y, the second with regard to x, we have

$$\left.\begin{aligned}\frac{\partial s}{\partial x}&=\left(\frac{d\theta}{dy}\right)+\frac{\partial\theta}{\partial\lambda}\frac{d\lambda}{dy}+\frac{\partial\theta}{\partial s}\frac{\partial s}{\partial y}\\ \frac{\partial s}{\partial y}&=\left(\frac{d\phi}{dx}\right)+\frac{\partial\phi}{\partial\lambda}\frac{d\lambda}{dx}+\frac{\partial\phi}{\partial s}\frac{\partial s}{\partial x}\end{aligned}\right\}\ldots\ldots\ldots\ldots(6);$$

and, on account of (4), these lead to

$$\left(\frac{d\theta}{dy}\right)+\frac{\partial\theta}{\partial\lambda}\frac{d\lambda}{dy}+\frac{\partial\theta}{\partial s}\left\{\left(\frac{d\phi}{dx}\right)+\frac{\partial\phi}{\partial\lambda}\frac{d\lambda}{dx}\right\}=0\ldots(7),$$

from which, since θ and ϕ are known functions, we derive s in the form

$$s=\sigma\left(x, y, z, p, q, \lambda, \frac{d\lambda}{dx}, \frac{d\lambda}{dy}\right),$$

and then from (5) we have r and t as functions of the same variables.

There are two conditions to be satisfied, namely

$$\frac{\partial r}{\partial y}=\frac{\partial s}{\partial x}\text{ and }\frac{\partial s}{\partial y}=\frac{\partial t}{\partial x},$$

which, on account of (4) and (7) reduce to one. Substituting in the first of equations (6) we obtain

$$\frac{d\sigma}{dx}-\frac{\partial\theta}{\partial\sigma}\frac{d\sigma}{dy}-\frac{\partial\theta}{\partial\lambda}\frac{d\lambda}{dy}-\left(\frac{d\theta}{dy}\right)=0\ldots\ldots\ldots(8),$$

which, since σ is a known function, is an equation, of the type indicated, to determine λ. When any value of λ has been determined from this equation, the result, on substitution in

$$r=\theta, \quad s=\sigma, \quad t=\phi,$$

leads to consistent values of r, s, and t, and hence by three quadratures to a value of z.

When the original equation (1) does not contain z, p, or q, we may regard λ as a function of x and y only; and when (1) does not contain x, y, or z, we may regard λ as a function of p and q only; but in general λ is a function of the five variables x, y, z, p, and q.

The expanded form of equation (8) is extremely complicated, but it belongs to the particular class of equations in which the characteristic invariant* is resoluble into two linear factors. The factors are

$$\left(\frac{du}{dx}\right)-\frac{\partial\theta}{\partial\sigma}\left(\frac{du}{dy}\right)=0,$$

$$\frac{\partial\sigma}{\partial\lambda_x}\left(\frac{du}{dx}\right)+\frac{\partial\sigma}{\partial\lambda_y}\left(\frac{du}{dy}\right)=0,$$

where
$$\lambda_x,\ \lambda_y=\frac{d\lambda}{dx},\ \frac{d\lambda}{dy}.$$

If equation (8) possesses an intermediate integral of the first order, it must clearly be of the form

$$u\,(x, y, z, p, q, \lambda, \sigma)=0;$$

and the equation itself must be the same as

$$\left(\frac{du}{dx}\right)+\frac{\partial u}{\partial\lambda}\frac{d\lambda}{dx}+\frac{\partial u}{\partial\sigma}\frac{d\sigma}{dx}=\frac{\partial\theta}{\partial\sigma}\left\{\left(\frac{du}{dy}\right)+\frac{\partial u}{\partial\lambda}\frac{d\lambda}{dy}+\frac{\partial u}{\partial\sigma}\frac{d\sigma}{dy}\right\}.$$

Comparing this with equation (8), and making use of (7), we readily find that u must be a common integral of the two equations

$$\frac{\partial u}{\partial\sigma}\Big/\frac{\partial u}{\partial\lambda}=\frac{\partial\theta}{\partial\sigma}\Big/\frac{\partial\theta}{\partial\lambda}+\frac{\partial\phi}{\partial\sigma}\Big/\frac{\partial\phi}{\partial\lambda},$$

$$\left\{\left(\frac{du}{\partial x}\right)-\frac{\partial\theta}{\partial\sigma}\left(\frac{du}{dy}\right)\right\}\Big/\frac{\partial u}{\partial\lambda}=\left(\frac{d\phi}{dx}\right)\Big/\frac{\partial\phi}{\partial\lambda}-\frac{\partial\theta}{\partial\sigma}\left(\frac{d\theta}{dy}\right)\Big/\frac{\partial\theta}{\partial\lambda}.$$

The conditions of co-existence of these equations appear to be the same as the conditions that (1) should possess an intermediate integral of the first order.

* Forsyth, *Theory of Differential Equations*, vol. vi., §§ 328 and 334.

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FACTORISATION OF $N=(Y^Y \mp 1)$ AND $(X^{XY} \mp Y^{XY})$.

By *Lt.-Col. Allan Cunningham, R.E.*, Fellow of King's College, London.

[The author is indebted to Mr. H. J. Woodall, A.R.C.Sc., for useful suggestions and for help in reading the proof-sheets.]

1. *Introduction.* This Paper is intended to give Rules for the factorisation of the four allied numbers

$$N=Y^Y-1, \qquad N'=Y^Y+1 \ldots\ldots\ldots\ldots(1),$$
$$N=X^{XY}\sim Y^{XY}, \quad N'=X^{XY}+Y^{XY}, [X \text{ prime to } Y].(2),$$

and to introduce the Tables (printed at the end of the Paper) of the factorisation thereof up to very high limits.

2. *Rarity of primes.* The salient property of these numbers is that they are *nearly all composite*, and are indeed nearly all algebraically resolvable.

In fact the only form among them not so resolvable is that of "Fermat's Numbers"—

$$N'=Y^Y+1, \text{ where } Y=2^e, \ [e=2^m] \ldots\ldots\ldots(3);$$

and, among these, the only known primes are the lowest two, $2^2+1=5$, and $4^4+1=257$, whilst the next two are known to be *composite*, viz.

$$16^{16}+1=274177.67280421310721;$$
$$256^{256}+1 \text{ has the factors* } (2^{13}.39+1)(2^{13}.119+1).$$

3. *Notation.* All symbols integers.

p, q denote odd *primes*.

ε denotes an *even* number; ω denotes an *odd* number.

$e=2^m$; s^2 denotes a square.

m denotes any factor of n.

$F(n)$, $F'(n)$, $\Phi(n)$, $\Phi'(n)$, see Art. 6; $\phi(n)$, $\phi'(n)$, see Art. 6.

4. *Algebraic and Arithmetical Factorisations.* The general process of factorisation of large numbers is naturally divided into two main, and very distinct, steps—

I. Algebraic. II. Arithmetical.

The algebraic resolution is described in Art. 5—16; the

* Discovered by the author.

arithmetical resolution is described in Art. 17–21f. The Factorisation Tables of N, N', which are the outcome of this Memoir, are described in Art. 22–22c.

5. *Algebraic Factors.* An algebraic function $f(x, y)$, which is an exact divisor of the algebraic expression $F(x, y)$ *for all values* of x, y is styled an *algebraic factor* of $F(x, y)$. If $f(x, y)$ itself has no such algebraic factors, or is in other words *irreducible*, it is styled an *algebraic prime factor* of $F(x, y)$, and is denoted (for shortness) by A.P.F. The maximum algebraic prime factor of $F(x, y)$ is denoted (for shortness) by M.A.P.F. These are denoted symbolically thus

$f(x, y)$ is *an* A.P.F. of $F(x, y)$..........................(4a),

$\phi(x, y)$ is *the* M.A.P.F. of $F(x, y)$...................(4b).

The most important property of these algebraic factors, in relation to factorisation, is that

$F(x, y) =$ the continued product of *all* its A.P.F. (including the M.A.P.F.)...(5).

[The arithmetical factors of the various A.P.F. are generally of quite different forms, involving different modes of search (described in Art. 17–21f)].

The resolution of $F(x, y)$ into its A.P.F. is therefore a most important—(usually the first)—step in the factorisation of large numbers.

6. *Algebraic Factors of Binomials.* The number and nature of the A.P.F. of Binomials $(x^n \mp y^n)$, where x, y have the *same exponent* (n), depend chiefly on the nature (prime or composite) of that index (n): so that a short notation exhibiting distinctly the relation to the exponent (n) is convenient. Let

$F(n) = x^n - y^n$, $\qquad F'(n) = x^n + y^n$ [x, y both $+$] ...(6a),

$f(n) =$ an A.P.F. of $F(n)$, $\qquad f'(n)$ an A.P.F. of $F'(n)$ ....(6b),

$\phi(n) =$ the M.A.P.F. of $F(n)$, $\qquad \phi'(n) =$ the M.A.P.F. of $F'(n)$...(6c).

Hereby

$F(1) = f(1) = \phi(1) = (x - y)$, $\qquad F'(1) = f'(1) = \phi'(1) = (x + y)$......(6$d$).

It will now be shown how to obtain the M.A.P.F. as the quotient of the products of various A.P.F. of $F(n)$, $F'(n)$. Here *five* types should be distinguished according to the form of n.

i. $n = e = 2^\mu$; ii. $n = \omega$; iii. $n = e\omega$; iv. $x = \xi^m$, $y = \eta^m$; v. $nxy = \square$.

7. TYPE i. $n = e = 2^\mu$; $[x \neq \xi^m,\ y \neq \eta^m]$.

$$F(e) = F(\tfrac{1}{2}e) \,||\, F'(\tfrac{1}{2}e); \qquad \phi(e) = F'(\tfrac{1}{2}e) \ldots\ldots (7a),$$
$$F(2) = F(1) \,||\, F'(1); \qquad \phi(2) = F'(1) \ldots\ldots (7b),$$
$$F(4) = F(1) \,|\, F'(1) \,||\, F'(2); \qquad \phi(4) = F'(2) \ldots\ldots (7c),$$
$$F(8) = F(1) \,|\, F'(1) \,|\, F'(2) \,||\, F'(4); \qquad \phi(8) = F'(4) \ldots\ldots (7d),$$
$$\vdots$$
$$F(e) = F(1) \,|\, F'(1) \,|\, F'(2) \,|\, F'(4) \,|\, \ldots\ldots \,||\, F'(\tfrac{1}{2}e) \ldots\ldots (7e),$$

and the factors $F'(e)$ are *irreducible* for all values of e, so that

$$\phi'(1) = F''(1),\ \ \phi'(2) = F''(2),\ \ \phi'(4) = F''(4),\ \ \ldots \phi'(e) = F''(e) \ldots\ldots (8).$$

[The single bars ($|$) used above are merely special multiplication symbols used to *separate distinctly* the various A.P.F. of $F(n)$; the double bar ($||$) is used to separate all the minor A.P.F. from the M.A.P.F. of $F(n)$. These symbols are most useful in arithmetical work: see the Tables on pages 72—74].

8. TYPE ii. $n = \omega$; $[x \neq \xi^m,\ y \neq \eta^m]$.

Let a, b, c denote unequal odd primes $[a < b < c]$.

The values of $\phi(n)$, $\phi'(n)$ are shown in the scheme below, for all the (odd) values of n required in this Memoir.

n	$\phi(n)$	$\phi'(n)$	
1	$F(1)$	$F'(1)$	...(9a),
a	$F(a) \div F(1)$	$F'(a) \div F'(1)$	...(9b),
a^2	$F(a^2) \div F(a)$	$F'(a^2) \div F'(a)$	...(9c),
a^α	$F(a^\alpha) \div F(a^{\alpha-1})$	$F'(a^\alpha) \div F'(a^{\alpha-1})$	...(9d),
ab	$\{F(ab).F(1)\} \div \{F(a).F(b)\}$	$\{F'(ab).F'(1)\} \div \{F'(a).F'(b)\}$	...(9e),
a^2b	$\{F(a^2b).F(a)\} \div \{F(a^2).F(ab)\}$	$\{F'(a^2b).F'(a)\} \div \{F'(a^2).F'(ab)\}$ ...	...(9f),
abc	$\dfrac{F(abc).F(a).F(b).F(c)}{F(bc).F(ca).F(ab).F(1)}$	$\dfrac{F'(abc)\ F'(a).F'(b).F'(c)}{F'(bc).F'(ca).F'(ab).F'(1)}$	...(g),

and the values of $F(n)$, $F'(n)$ are shown in the scheme below, expressed as the continued product of their A.P.F., for the same values of n as above: the A.P.F. being arranged in order of magnitude (the smallest on the left).

n	$F(n) = \Pi\{f(n)\}$	$F'(n) = \Pi\{f'(n)\}$
1	$\phi(1)$	$\phi'(1)$ .... (10a),
a	$\phi(1).\phi(a)$	$\phi'(1).\phi'(a)$ .... (10b),
a^2	$\phi(1).\phi(a)\ \phi(a^2)$	$\phi'(1).\phi'(a).\phi'(a^2)$ .... (10c),
a^α	$\phi(1).\phi(a).\phi(a^2)\ldots\phi(a^{\alpha-1})$	$\phi'(1).\phi'(a).\phi'(a^2)\ldots\phi'(a^{\alpha-1})$ .... (10d),
ab	$\phi(1).\phi(a).\phi(b).\phi(ab)$	$\phi'(1)\ \phi'(a).\phi'(b).\phi'(ab)$ .... (10e),
a^2b	$\phi(1).\phi(a).\phi(a^2).\phi(b).\phi(ab).\phi(a^2b)$	$\phi'(1).\phi'(a).\phi'(a^2).\phi'(b)\ \phi'(ab).\phi'(a^2b)$.(10f),
abc	$\phi(1).\phi(a).\phi(b).\phi(c).$ $\phi(bc).\phi(ca).\phi(ab).\phi(abc)$	$\phi'(1).\phi'(a).\phi'(b).(\phi'(c)$ $\phi'(bc).\phi'(ca).\phi'(ab).\phi'(abc)$...(10g).

Comparing the above formulæ, it is seen that, when $n = \omega$,

$$\phi(n),\ \phi'(n) \text{ are of the same form} \ldots\ldots (11a),$$
$$F(n),\ F'(n) \text{ are of the same form} \ldots\ldots (11b).$$

9. TYPE iii. $n = e\omega = 2^\mu . \omega$; $[x \neq \xi^m,\ y \neq \eta^m]$.

The reduction of $F(n)$, $F'(n)$ is treated separately in Art. 9*a*, 9*b*.

9*a*. TYPE iii*a*. $n = e\omega$, $[x \neq \xi^m,\ y \neq \eta^m]$. *Reduction of* $F(n)$.

The function $F(n)$ may be first resolved, as far as the quantity $e = 2^\mu$ is concerned, in a way similar to that used for $F(e)$ in Art. 7, and with similar notation; thus

$$F(e\omega) = F(\tfrac{1}{2}e\omega) \,||\, F'(\tfrac{1}{2}e\omega); \qquad \phi(e\omega) = \phi'(\tfrac{1}{2}e\omega) \ldots\ldots (12a),$$
$$F(2\omega) = F(\omega) \,||\, F'(\omega); \qquad \phi(2\omega) = \phi'(\omega) \ldots\ldots (12b),$$
$$F(4\omega) = F(\omega) \,|\, F'(\omega) \,||\, F'(2\omega), \qquad \phi(4\omega) = \phi'(2\omega) \ldots\ldots (12c),$$
$$F(8\omega) = F(\omega) \,|\, F'(\omega) \,|\, F'(2\omega) \,||\, F'(4\omega); \qquad \phi(8\omega) = \phi'(4\omega) \ldots\ldots (12d),$$
$$\vdots$$
$$F(e\omega) = F(\omega) \,|\, F(\omega) \,|\, F'(2\omega) \,|\, F'(4\omega) \,|\, \ldots \,||\, F'(\tfrac{1}{2}e\omega) \ldots\ldots (12e).$$

The factors $F(\omega)$, $F'(\omega)$ above are of the Type ii. of Art. 8. This A.P.F. may be found by the Rules of that Article; and they may then be expressed as the continued product of their A.P.F. by the Rules of that Article.

[The use of the single bar (|) and double bar (||) is similar to that explained at foot of Art. 7].

9*b*. TYPE iii*b*. $n = e\omega$; $[x \neq \xi^m,\ y \neq \eta^m]$. *Reduction of* $F'(n)$.

Here $F'(e\omega)$ is *irreducible*, as far as the factor $e = 2^\mu$ of the exponent is concerned (compare Art. 7). The reduction of the factor ω is similar to that of the $n = \omega$ in Art. 8. The value of $\phi'(n)$ is shown in the scheme below for all the values of n required in this Memoir: those of $F'(n)$ are also shown alongside, expressed as the continued product of their A.P.F.; the A.P.F. being arranged in order of magnitude (the smallest on the left).

n	$\phi'(n)$	$F'(n)$	
$2a$	$F'(2a) \div F'(2)$	$\phi'(2).\phi'(2a)$	...(13*a*),
$4a$	$F'(4a) \div F'(4)$	$\phi'(4).\phi'(4a)$	...(13*b*),
ea	$F'(ea) \div F'(e)$	$\phi'(e).\phi'(ea)$	...(13*c*),
$2a^2$	$F'(2a^2) \div F'(2a)$	$\phi'(2).\phi'(2a).\phi'(2a^2)$	...(13*d*),
$4a^2$	$F'(4a^2) \div F'(4a)$	$\phi'(4).\phi'(4a)\ \phi'(4a^2)$	...(13*e*),
$2ab$	$\dfrac{F'(2ab).F'(2)}{F'(2a).F'(2b)}$	$\phi'(2).\phi'(2a).\phi'(2b).\phi'(2ab)$	...(13*f*),
$2a^2b$	$\dfrac{F'(2a^2b).F'(2a)}{F'(2a^2).F'(2ab)}$	$\phi'(2).\phi'(2a).\phi'(2a^2).\phi'(2b).\phi'(2ab)\ \phi'(2a^2b)$...	...(13*g*).

10. TYPE iv. $x = \xi^m$, $y = \eta^m$.

When x, y, the bases appearing in $F(n)$, $F'(n)$, are themselves both m^{th} powers of smaller bases ξ, η—a case excluded

from Art. 7, 8, 9—then the A.P.F. of $F(n)$, $F'(n)$, viewed as functions of x, y, are *further reducible*, as functions of ξ, η. For, writing Φ, Φ' as functional symbols of ξ, η, when x, y are changed to ξ^m, η^m, then

$$F(n) = x^n \sim y^n = \xi^{mn} \sim \eta^{mn} = \Phi(mn) \quad \text{.......................(14a)},$$
$$F'(n) = x^n + y^n = \xi^{mn} + \eta^{mn} = \Phi'(mn) \quad \text{.......................(14b)},$$

and the A.P.F. of $\Phi(mn)$, $\Phi'(mn)$ may now be found by the Rules of Art. 7, 8, 9, where mn now takes the place of the n of those Articles. And $\Phi(mn)$, $\Phi'(mn)$ may also be expressed as the continued product of their A.P.F. by the Rules of those Articles.

11. TYPE V. $n\,xy = \square$.

The development of this case (for factorisation purposes) depends on the expression of the M.A.P.F. of $F(n)$, $F'(n)$, *i.e.* of $\phi(n)$, $\phi'(n)$ in one or other of the *impure** 2^{ic} forms, $(P^2 \mp n\,xy\,Q^2)$, or in some derivative thereof $(P^2 \mp m\,xy\,Q^2)$ where m is some factor of n, *in the general case* (*i.e.* independently of the condition $n\,xy$, or $m\,xy = \square$). The necessary and sufficient condition for this is

$$n = \omega, \text{ or } 2\omega \text{ only}, \quad [n \neq 4m] \quad \text{...........................(15)}.$$

A preliminary discussion of the general case occupies Art. 12, 12*a*–*d*. The application to factorisation occupies Art. 15.

12. *Impure* 2^{ic} *Forms* $(n \neq ms^2)$. When n has the form

$$n = \omega, \text{ or } 2\omega, \text{ and } \neq ms^2 \quad \text{...............................(16)},$$

the M.A.P.F. $\phi(n)$, $\phi'(n)$ are always (algebraically) expressible in one or other of the *impure* 2^{ic} forms as below—

n	$\phi(n)$	$\phi'(n)$	
$4i+1$	$P^2 - nxyQ^2$	$P^2 + nxyQ^2$	....................(17*a*),
$4i-1$	$P^2 + nxyQ^2$	$P^2 - nxyQ^2$	....................(17*b*),
$e\omega$	.	$P^2 + nxyQ^2$	....................(17*c*),

where P, Q are certain functions of n, x, y explained in Art. 13 and tabulated up to $n = 46$ in the Table A therewith.

12*a*. *Impure* 2^{ic} *Forms* (*n composite*).

If n be as in Art. 12, but composite, *i.e.* if

$$n = \omega, \text{ or } 2\omega = km, \text{ and } \neq \mu s^2, \quad [k = \omega, \text{ and prime to } m] \quad \text{.........(18)},$$

then $\phi(n)$, $\phi'(n)$, the M.A.P.F. of $F(n)$, $F'(n)$ are always (algebraically) expressible not only in one or other of the *impure* 2^{ic} forms $(P^2 \mp n\,xy\,Q^2)$ as in Art. 12, but also in one or other of each of the derivatives thereof,

$$(P^2 \mp m\,xy\,Q^2), \quad (P^2 \mp k\,xy\,Q^2),$$

* In this Memoir $(P^2 \mp n\,xyQ^2)$ is styled an *impure* 2^{ic} form; and $(P^2 \mp nQ^2)$ is styled a *pure* 2^{ic} form when n does not depend on x, y, ξ, η, &c.

similar to (17*a.b,c*), where m, k now take the place of n in those formulæ, the sign ($\mp$) being determined by the forms of m or k: and this is possible for every way in which n can be resolved into two cofactors ($n = km$): but the P, Q are *quite different*.

The process of finding the P, Q in this case from those of Art. 12 is rather troublesome. One way of effecting it is shown below, for the case of $\phi(n) = (P^2 \mp m\,xy\,Q^2)$: it depends on expressing $\phi(n)$ as a quotient of two 2^{ic} forms *of same determinant* ($D = \pm m\,xy$). The case of $\phi'(n)$ is precisely similar.

Write $x^k = \xi$, $y^k = \eta$, $[k = \omega]$, whereby $F(n)$ becomes a function of ξ, η, which may be written $\Phi_k(m)$, with $\phi_k(m)$ as its M.A.P.F. Then, by Art. 8,

$$\phi(n) = \frac{F(km) \,.\, F(1)}{F(k) \,.\, F(m)} = \frac{\Phi_k(m)}{\Phi_k(1)} \cdot \frac{F(1)}{F(m)} = \frac{\phi_k(m)}{\phi_1(m)},$$

and, by Art. 12,

$$\phi_k(m) = \text{one of } (P_k^2 \mp m\,\xi\eta\,Q_k^2),$$
$$= \text{one of } (P_k^2 \mp m\,xy\,Q_k'^2),$$

where $$Q_k' = (xy)^{\frac{1}{2}(k-1)} \,.\, Q_k \ldots\ldots [k \text{ being odd}],$$

and $$\phi_1(m) = \text{one of } (P_1^2 \mp m\,xy\,Q_1^2)$$

therefore $$\phi(n) = \text{one of } \frac{P_k^2 \mp mxyQ_k'^2}{P_1^2 \mp mxyQ_1^2}, \quad [\text{same sign in both}] \ldots\ldots (19).$$

As these 2^{ic} forms are *of same determinant* ($D \pm m\,xy$), their quotient $\phi(n)$ is (algebraically) expressible in the reduced forms,

$$\phi(n) = P^2 - mxyQ^2 \ldots\ldots\ldots\ldots\ldots\ldots (19a),$$

by the process of *conformal division*.*

Here P_k, Q_k and P_1, Q_1 are given by the Rules of Art. 12. The carrying through of this whole process is rather troublesome where n is large.

12*b*. *Impure 2^{ic} Forms of Sub-Factors* ($n = km$):

When n is composite, as in (18), every A.P.F. of $F(n)$, $F'(n)$, say $\phi(k)$, $\phi(m)$, $\phi'(k)$, $\phi'(m)$ is also (algebraically) expressible in its own *impure* 2^{ic} *forms* of determinant $D = \pm k\,xy$, $m\,xy$ under Rules similar to those of Art. 12, 12*a*. Here k, m take the place of n of these Articles.

12*c*. *Impure 2^{ic} Forms.* $n = ms^2$, [no square factor in m].

When $n = ms^2$; then, by a suitable substitution for x, y,

* *i.e.* division *with preservation of 2^{ic} form.* See the author's Paper on *Connexion of Quadratic Forms* in *Proc. Lond. Math. Soc.*, vol. xxviii., 1897, pp. 289 *et seq.* for a full explanation of the process.

the quantities $F(n)$, $F'(n)$, which are functions of x, y, may be brought to the forms $\Phi(m)$, $\Phi'(m)$, where Φ, Φ' are functions of ξ, η (and m is free from squares). Hereby $\Phi(m)$, $\Phi'(m)$ and their A.P.F. now fall under the forms of Art. 12*a*.

The only cases required for this Paper are

TYPE i. $n = a^2, a^3, a^4, \ldots$; TYPE ii. $n = a^2 b$; TYPE iii. $n = 2a^2, 2a^3$.

The reduction of $F(n)$, $F'(n)$ to the forms $\Phi(n)$, $\Phi'(n)$ is here first shown for each type and followed by the expression of $\phi(m)$, $\phi'(m)$ in the form $(P^2 \mp m\,\xi\eta\,Q^2)$:

TYPE i. $n = a^2, a^3, a^4, \ldots a^\alpha$. Take $\xi = x^{n \div a}$, $\eta = y^{n \div a}$,

$$n = a^\alpha \text{ gives } F(n) = x^n \sim y^n = \xi^a \sim \eta^a = \Phi(a)\,;\quad m = a \ldots\ldots\ldots (20a),$$

$$F'(n) = x^n + y^n = \xi^a + \eta^a = \Phi'(a)\,;\quad m = a \ldots\ldots\ldots (20b).$$

TYPE ii. $n = a^2 b$. Take $\xi = x^a$, $\eta = y^a$.

$$n = a^2 b \text{ gives } F(n) = x^n \sim y^n = \xi^{ab} \sim \eta^{ab} = \Phi(ab)\,;\quad m = ab \ldots\ldots (21a),$$

$$F'(n) = x^n + y^n = \xi^{ab} + \eta^{ab} = \Phi'(ab)\,;\quad m = ab \ldots\ldots (21b).$$

TYPE iii. $n = 2a^2, 2a^3, \ldots 2a^\alpha$. Take $\xi = x^{n \div 2a}$, $n = y^{n \div 2a}$,

$$n = 2a^\alpha \text{ gives } F'(n) = x^n + y^n = \xi^{2a} + \eta^{2a} = \Phi'(2a)\,;\quad m = 2a \ldots\ldots (23).$$

And the reduced function $\phi(m)$, $\phi'(m)$ now fall under Art. 12 (since m is free from square factors), so that the A.P.F. $\phi(m)$, $\phi'(m)$ may now be expressed in the form $(P^2 \mp m\,\xi\eta\,Q^2)$ as in the scheme below. Here P, Q are the same functions of ξ, η as shown in Art. 12 for x, y.

Type	n	ξ	η	m	$\phi(m)$	$\phi'(m)$
i.	a^2	x^a	y^a	$a = 4i+1$	$P^2 - m\xi\eta Q^2$	$P^2 + m\xi\eta Q^2 \ldots (23a)$,
				$a = 4i-1$	$P^2 + m\xi\eta Q^2$	$P^2 - m\xi\eta Q^2 \ldots (23b)$,
i.	a^α	$x^{a \div a}$	$y^{a \div a}$	$a = 4i+1$	$P^2 - m\xi\eta Q^2$	$P^2 + m\xi\eta Q^2 \ldots (24a)$,
				$a = 4i-1$	$P^2 + m\xi\eta Q^2$	$P^2 - m\xi\eta Q^2 \ldots (24b)$,
ii.	$a^2 b$	x^a	y^a	$ab = 4i+1$	$P^2 - m\xi\eta Q^2$	$P^2 + m\xi\eta Q^2 \ldots (25a)$,
				$ab = 4i-1$	$P^2 + m\xi\eta Q^2$	$P^2 - m\xi\eta Q^2 \ldots (25b)$,
iii.	$2a^2$	x^a	y^a	$2a$	.	$P^2 - m\xi\eta Q^2 \ldots (26)$.

12*d*. *Examples.* A few examples will serve to illustrate the principles of Art. 12, 12*a*,*b*. The quantities P, Q, are of course *different in each* of the 2ic forms here shown.

Ex. 1°. $F(15)$, $F'(15)$ have $\phi(15)$, $\phi'(15)$ as M.A.P.F., and have also $\phi(5)$, $\phi(3)$, $\phi(1)$; $\phi'(5)$, $\phi'(3)$, $\phi'(1)$ as lower A.P.F.

$$\phi(15) = P^2 + 15xyQ^2 = P^2 - 5xyQ^2 = P^2 + 3xyQ^2\,;$$
$$\phi'(15) = P^2 - 15xyQ^2 = P^2 + 5xyQ^2 = P^2 - 3xyQ^2\,;$$
$$\phi(5) = P^2 - 5xyQ^2\,;\quad \phi(3) = P^2 + 3xyQ^2\,;$$
$$\phi'(5) = P^2 + 5xyQ^2\,;\quad \phi'(3) = P^2 - 3xyQ^2.$$

Ex. 2°. $F'(30)$ has $\phi'(30)$ as *M.A.P.F.*, and has $\phi'(10)$, $\phi'(6)$, $\phi'(2)$ as lower A.P.F.

$$\phi'(30) = P^2 - 30xyQ^2 = P^2 - 10xyQ^2 = P^2 - 6xyQ^2 = P^2 - 2xyQ^2,$$
$$\phi'(10) = P^2 - 10xyQ^2,\quad \phi'(6) = P^2 - 6xyQ^2,\quad \phi'(2) = P^2 - 2xyQ^2.$$

Ex. 3°. $F(45)$, $F'(45)$ have $\phi(45)$, $\phi'(45)$ as M.A.P.F., and have also $\phi(15)$, $\phi(9)$, $\phi(5)$, $\phi(3)$, $\phi(1)$; $\phi'(15)$, $\phi'(9)$, $\phi'(5)$, $\phi'(3)$, $\phi'(1)$ as lower A P.F.

$$\phi(45) = P^2 - 5xyQ^2 = P^2 + 3xyQ^2 = P^2 + 15xyQ^2;$$

$$\phi'(45) = P^2 + 5xyQ^2 = P^2 - 3xyQ^2 = P^2 - 15xy^2;$$

$\phi(15)$, $\phi(5)$, $\phi(3)$ and $\phi'(15)$, $\phi'(5)$, $\phi'(3)$ have the forms shown in Ex. 1°.

$$\phi(9) = P^2 + 3xyQ^2, \quad \phi'(9) = P^2 - 3xyQ^2.$$

13. *Values of P, Q.* The quantities of P, Q are homogeneous symmetric functions of x, y so that they may be written

$$P = A_0x^{\varpi} + A_1x^{\varpi-1}y + A_2x^{\varpi-2}y^2 + \ldots + A_1xy^{\varpi-1} + A_0y^{\varpi} \ldots\ldots (27a),$$

$$Q = B_0x^{\kappa} + B_1x^{\kappa-1}y + B_2x^{\kappa-2}y^2 + \ldots + B_1xy^{\kappa-1} + B_0y^{\kappa} \ldots\ldots (27b).$$

Here, using $\tau(z)$ to denote the *Totient* of z, then—

$$n = \omega \text{ has } \varpi = \tfrac{1}{2}\tau(\omega), \quad \kappa = \varpi - 1 \ldots\ldots (28a),$$

$$n = 2\omega \text{ has } \varpi = \tfrac{1}{2}\tau(2\omega), \quad \kappa = \varpi - 1 \ldots\ldots (28b).$$

And A_r, B_r are functions of n only, (not of x, y)............(29a),

also $A_0 = 1$, $B_0 = 1$ *always*............(29b).

$A_1 = \frac{1}{2}(n+1)$, when $n = 4i+1 = p$; $A_1 = \frac{1}{2}(n-1)$, when $n = 4i-1 = pq$..(29c),

$A_1 = \frac{1}{2}(n-1)$, when $n = 4i-1 = p$; $A_1 = \frac{1}{2}(n+1)$, when $n = 4i+1 = pq$..(29d),

$A_1 = \frac{1}{2}n$, when $n = 2\omega$............(29e).

The rest of the coefficients (A_r, B_r) are complicated functions of n, and are difficult* to calculate, so that (for the present purpose) they are most conveniently taken from Tables.

Table† A on page 57 gives the values of the coefficients A_r, B_r which appear in the terms

$$A_rx^{\varpi-r}y^r, \quad A_rx^ry^{\varpi-r}, \quad B_rx^{\kappa-r}y^r, \quad B_rx^ry^{\kappa-r}. \ldots\ldots (30),$$

for all the values of the index n up to $n = 46$ (except $n = 43$), the middle coefficient—(or the middle pair of coefficients, if equal)—being enclosed in brackets. They are tabulated *in the same order* as in the formulæ 27a, b, so that it should be easy to allocate them to those terms, as the Table gives the value of ϖ, κ also in the same line.

The Table is drawn up really for those values of $\phi(n)$, $\phi'(n)$ which are expressible in the form

$$\phi(n) \text{ or } \phi'(n) = P^2 - D.Q^2 \ldots\ldots (31),$$

where $D = nxy$, as in Art. 12a,b or $= n\xi n$, as in Art. 12c,

this being the form required for the important Aurifeuillian notation (Art. 15).

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* The method of calculation is explained in Ed. Lucas's Memoir *Sur les formules de Cauchy et de Lejeune Dirichlet* in the Report of the *Association Française pour l'Avancement des Sciences*, Congress of Paris, 1878.

† This Table is extracted (with some alterations and corrections by the present author) partly from Ed. Lucas's Memoir *Sur la Série récurente de Fermat*, Rome, 1879, and partly (with some changes) from the Memoir above quoted. Both these Memoirs are out of print, and difficult to obtain: (see Appendix II. for errata in the originals).

TAB. A.

n	ϕ, ϕ'	ϖ	*Coefficients* A_r *in* P	κ	*Coefficients* B_r *in* Q
2	ϕ'	1	1,1	0	1
3	ϕ'	1	1,1	0	1
5	ϕ	2	1,(3),1	1	1,1
6	ϕ'	2	1,(3),1	1	1,1
7	ϕ'	3	1,(3,3),1	2	1,(1),1
10	ϕ'	4	1,5,(7),5,1	3	1,(2,2),1
11	ϕ'	5	1,5,(−1,−1),5,1	4	1,1,(−1),1,1
13	ϕ	6	1,7,15,(19),15,7,1	5	1,3,(5,5),3,1
14	ϕ'	6	1,7,3,(−7),3,7,1	5	1,2,(−1,−1),2,1
15	ϕ'	4	1,8,(13),8,1	3	1,3,3,1
17	ϕ	8	1,9,11,−5,(−15),−5,11,9,1	7	1,3,1,(−3,−3),1,3,1
19	ϕ'	9	1,9,17,27,(31,31),27,17,9,1	8	1,3,5,7,(7),7,5,3,1
21	ϕ	6	1,10,13,(7),13,10,1	5	1,3,(2,2),3,1
22	ϕ'	10	1,11,27,33,21,(11),21,33,27,11,1	9	1,4,7,6,(3,3),6,7,4,1
23	ϕ	11	1,11,9,−19,−15,(25,25), −15,−19,9,11,1	10	1,3,−1,−5,1,(7),1, −5,−1,3,1
26	ϕ'	12	1,13,19,−13,−11,13,(7), 13,−11,−13,19,13,1	11	1,4,1,−4,1,(2,2),1, −4,1,4,1
29	ϕ	14	1,15,33,13,15,57,45,(19), 45,57,15,13,33,15,1	13	1,5,5,1,7,11,(5,5) 11,7,1,5,5,1
30	ϕ'	8	1,15,38,45,(43),45,38,15,1	7	1,5,8,(8,8),8,5,1
31	ϕ'	15	1,15,43,83,125,151,169,(173,173), 169,151,125,83,43,15,1	14	1,5,11,19,25,29,31,(31), 31,29,25,19,11,5,1
33	ϕ	10	1,16,37,19,−32,(−59),−32,19,37,16,1	9	1,5,6,−1,(−9,−9),1,6,5,1
34	ϕ'	16	1,17,59,119,181,221,243,255,(257) 255,243,221,181,119,59,17,1	15	1,6,15,26,35,40,43,(44, 44),43,40,35,26,15,6,1
35	ϕ'	12	1,18,48,11,−55,−11,(47),−11, −55,11,48,18,1	11	1,6,7,−5,−8,(5,5), −8,−5,7,6,1
37	ϕ	18	1,19,79,183,285,349,397,477,579,(627), 579,477,397,349,285,183,79,19,1	17	1,7,21,39,53,61,71,87, (101,101),87,71,61,53, 39,21,7,1
38	ϕ'	18	1,19,47,−19,−135,−57,179,209, −83,(−285),−83,209,179,−57, −135,−19,47,19,1	17	1,6,5,−14,−21,10,39,14, (−37,−37),14,39,10, −21,−14,5,6,1
39	ϕ'	12	1,20,73,119,142,173,(193), 173,142,119,73,20,1	11	1,7,16,21,25,(30,30), 25,21,16,7,1
41	ϕ	20	1,21,67,49,7,35,15,11,−23,−65,(−31), −65,−23,11,15,35,7,49,67,21,1	19	1,7,11,3,3,5,1,1,−9,(−7, −7),−9,1,1,5,3,3,11,7,1
42	ϕ'	12	1,21,74,105,55,−42,(−91), −42,55,105,74,21,1	11	1,−13,−45,68,83,(−120, −120),83,68,−45,−13,1
46	ϕ'	22	1,23,103,253,469,759 1131,1541,1917, 2231,2463,(2553),2463,2231,1917, 1541,1131,759,469,253,103,23,1	21	1,8,25,52,89,138,197,256, 307,348,(373,373),348,307, 256,197,138,89,52,25,8,1

14. *Sum of coefficients* A_r, B_r.

Let $\Sigma(A)$, $\Sigma(B)$ denote the sums of the coefficients in P, Q.

Let (τ', v'), (τ, v)* be solutions of

$$\tau'^2 - nv'^2 = -1, \quad \tau^2 - nv^2 = +1 \qquad (31a),$$

whereby
$$(nv')^2 - n\tau'^2 = n \qquad (31b).$$

Now take $x = 1$, $y = 1$. These values reduce P, Q to

$$P = \Sigma(A), \quad Q = \Sigma(B) \qquad (32),$$

whereby
$$\phi(n) \text{ or } \phi'(n) = P^2 - nQ^2 = \{\Sigma(A)\}^2 - n\{\Sigma(B)\}^2 \qquad (33),$$

But $\phi(n)$ or $\phi'(n)$ are at same time reduced to either n or 1, as follows:—

i. $n = 4i + 1 = p$ gives $\phi(n) = \dfrac{1^n - 1^n}{1 - 1} = n$,

whence
$$\Sigma(A) = nv', \quad \Sigma(B) = \tau' \qquad (34a).$$

ii. $n = 4i + 1 = pq$ gives $\phi(n) = \dfrac{(1^m - 1^m)(1^1 - 1^1)}{(1^p - 1^p)(1^q - 1^q)} = 1$,

whence
$$\Sigma(A) = \tau, \quad \Sigma(B) = v \qquad (34b).$$

iii. $n = 4i - 1 = p$ gives $\phi'(n) = 1$, *always*,

whence
$$\Sigma(A) = \tau, \quad \Sigma(B) = v \qquad (34c),$$

iv. $n = 2i = 2p$ gives $\phi'(m) = 1$, *always*,

whence
$$\Sigma(A) = \tau, \quad \Sigma(B) = v \qquad (34d).$$

These Results ($34a$–d), giving the values of $\Sigma(A)$, $\Sigma(B)$ in terms of the known solutions (τ', v'), (τ, v) of ($31a, b$), are very useful—(being easily applied)—*tests*† *of the correctness* of the tabulated coefficients (A_r, B_r).

A Table is subjoined giving the values of $\phi(n)$ or $\phi'(n)$, $\Sigma(A)$, $\Sigma(B)$—as in the formulæ ($34a$–d)—for ready application of the Test, for all the values of n in the general Table of A_r, B_r (Table A).

n	2	3	5	6	7	10	11	13	14	15*	17	19	21	22	23	26	29
ϕ, ϕ'	2	1	5	1	1	1	1	13	1	1	17	1	1	1	1	1	29
$\Sigma(A)$	2	2	5	5	8	19	10	65	15	31	17	170	55	197	24	51	377
$\Sigma(B)$	1	1	2	2	3	6	3	18	4	8	4	39	12	42	5	10	70

n	30*	31	33	34	35*	37*	39*	41	42*	46
ϕ, ϕ'	1	1	1	1	1	37	1	41	1	1
$\Sigma(A)$	241	1520	23	2149	71	5365	1249	205	337	24335
$\Sigma(B)$	44	273	4	420	12	882	200	32	52	3588

* This symbol τ, as here used, must not be confused with its use as the symbol for "Totient," as used in Art. 13.

† In fact it was by this Test that several Errata in each of Lucas's printed Tables were discovered. A List of these is given in Appendix II.

In most of the cases the minimum values of (τ', υ'), (τ, υ) are the ones to be used in the formulæ (34*a*–*d*) for $\Sigma(A)$, $\Sigma(B)$, but, in the cases marked * the solutions next greater than the minimum (τ', υ'), (τ, υ) are the ones required, viz.

n; $\tau^2 - n\upsilon^2 = +1$	m; $\tau'^2 - n\upsilon'^2 = -1$	m; $\tau^2 - n\upsilon^2 = +1$
15; $31^2 - 15.8^2 = +1$	37; $882^2 - 37.145^2 = -1$	30; $241^2 - 30.44^2 = +1$
35; $71^2 - 35.12^2 = -1$		42; $337^2 - 42.52^2 = +1$
39; $1249^2 - 39.200^2 = +1$		

15. *Aurifeuillians.* The formulæ of Art. 12–12*c* show that when the determinant $(\pm D)$ of the *impure* 2^{ic} forms (17*a*–*c*, 19*a*, 23*a*–26) has the form

$$\pm D = nxy \text{ or } mxy = s^2, \text{ a perfect square} \quad \text{(35)},$$

so that one or other of the R.P.F., *i.e.* $\phi(n)$, $\phi'(n)$, [or $\phi(m)$, $\phi'(m)$] is (algebraically) expressible as a *difference of squares*, viz.

$$\text{i. } n, \text{ or } m = 4i + 1 \text{ gives } \phi(n \text{ or } m) = P^2 - (sQ)^2 \quad \text{(36a)},$$

$$\text{ii. } n, \text{ or } m = 4i - 1 \text{ gives } \phi'(n \text{ or } m) = P^2 - (sQ)^2 \quad \text{(36b)},$$

$$\text{iii. } n, \text{ or } m = 2\omega \text{ gives } \phi'(n \text{ or } m) = P^2 - (sQ)^2 \quad \text{(36c)},$$

and is therefore at once (algebraically) resolvable into two cofactors (say L, M) so that

$$\phi(n \text{ or } m), \text{ or } \phi'(n \text{ or } m) = L.M \quad \text{(37a)},$$

where
$$L = P - sQ, \quad M = P + sQ \quad \text{(37b)}.$$

The functions $\phi(n \text{ or } m)$, $\phi'(n \text{ or } m)$, which are resolvable in this way, are styled* *Aurifeuillians*, and are described as of† order n or m. The two algebraic co-factors are styled *Aurifeuillian Factors.*

The condition of this resolution (35), styled the *Aurifeuillian condition*, may be satisfied in the following ways:

1°. When $y = 1$, so that $F(n) = y^n - 1$, $F'(n) = y^n + 1$,

then $y = n\eta^2$ gives $D = (n\eta)^2 = s^2$ ...(38*a*),

2°. When x and $y > 1$; then $x = \xi^2$, $y = n\eta^2$;

or $x = n\xi^2$, $y = \eta^2$; give $D = (n\xi\eta)^2 = s^2$ ...(38*b*),

---

* From having been first tabulated, and used for factorisation, by M. Aurifeuille of Toulouse; see Lucas's Mémoirs above quoted. They are obviously of great use in factorisation: thus nearly all numbers $N' = Y^Y + 1$, in which $Y \neq 4n$, are of this kind, or else some of their algebraic factors are of this kind.

† Special names are applied to distinguish the orders. Thus those of order 2, 3, 5, 6, &c., are styled *Bin-*, *Trin-*, *Quint-*, *Sext-*, &c.—*Aurifeuillians.* These names are due to the present author, who has made a special study of them, see his Papers in *Lond. Math. Soc. Proc.*, vol. xxix., 1898, and *Messenger of Mathematics*, vol. xxxix., 1909, and many places in the *Educational Times Reprints.*

3. When x and $y>1$; and $n=n_1n_2$, then $x=n_1\xi^2$, $y=n_2\eta^2$;

or $x=n_2\xi^2$, $y=n_1\eta^2$; give $D=(n\xi\eta)^2=s^2$...(38c).

[In the formulæ (38*a*, *b*, *c*) m may be substituted for n, when required].

It will be seen (Art. 12, 12*c*, 15) that Aurifeuillians can only occur among the A.P.F. of $F(n)$, $F'(n)$; and that, when n is *composite* $(n=km)$, the M.A.P.F. $\phi(m)$, $\phi'(m)$ may have an Aurifeuillian of any order (k, m) for which the Aurifeuillian condition (35) is satisfied; but, *for particular values of* x, y, this condition can only be satisfied in one way. Similarly, when n is *composite* $(n=km)$, the lower A.P.F. of $F(n)$, $F'(n)$, say $\phi(m)$, $\phi'(m)$ may possibly each have one Aurifeuillian form (of order m) for particular values of x, y.

15*a*. *Tabulation.* In the Factorisation Tables at end of this Paper the Aurifeuillians are recognisable at sight, because their Aurifeuillian Factors (L, M) are shown separated by a colon (:), thus

$$\phi(m) \text{ or } \phi'(m)=L:M \qquad \text{...(39).}$$

The *order* (m or m') of Aurifeuillians $\phi(m)$, $\phi'(m)$ occurring in each $F(n)$ or $F'(n)$ is shown by the figures in the columed headed *Aur.*

Ex. 1. $n=27$. $N'=27^{27}+1=3^{81}+1$.

Here $N'=\phi'(1)\phi'(3).\phi'(9).\phi'(27).\phi'(81)$, by Art 12*c*.

And each of the A.P.F. after $\phi'(1)$ is a *Trin-Aurifeuillian* ($m'=3$, Art. 15).

Ex. 2. $n=45$. $N=45^{45}-1$.

Here $N=\phi(1).\phi(3).\phi(5).\phi(9).\phi(15).\phi(45)$.

And each of the algebraic factors $\phi(5)$, $\phi(15)$, $\phi(45)$ is a *Quint-Aurifeuillian*, ($m=5$, see Art. 12*d*, 15); but $\phi(3)$, $\phi(9)$ are not Aurifeuillians.

Exceptional Cases, $(L=1)$. In a few cases, when x, y are small, the lesser Aurifeuillian factor (L) reduces to $L=1$. In these cases the number N or N' may be written N or $N'=1:M$, to show that N or N' is the limiting case of an Aurifeuillian.

Ex. $x=2$; $N'=1^2+(2.1^2)^2=1:5$; $[L=1, M=5]$.

$x=3$; $N'=\{1^3+(3.1^2)^3\}\div(1+3.1^2)=1:7$; $[L=1, M=7]$.

16. *Quotient-Aurifeuillians.*

If $A_1=L_1M_1$ and $A_2=L_2M_2$ be Aurifeuillians of *same order* (m), and A_1 be a divisor of A_2 with quotient $\mathfrak{A}$, thus

$$\mathfrak{A}=A_2\div A_1 \text{ is an Aurifeullian of same order } (m) \qquad \text{...(40),}$$

and

$$\mathfrak{A}=\mathfrak{L}.\mathfrak{M}=\frac{A_2}{A_1}=\frac{L_2M_2}{L_1M_1} \qquad \text{...(40a),}$$

and the co-factors $\mathfrak{L}$, $\mathfrak{M}$ of $\mathfrak{A}$ can be found directly by the preceding Rules, or by the property of such Quotients, viz.

$$\mathfrak{L} = \frac{L_2}{L_1} \text{ or } \frac{L_2}{M_1}, \quad \mathfrak{M} = \frac{M_2}{M_1} \text{ or } \frac{M_2}{L_1} \dots\dots\dots\dots\dots\dots(41).$$

These divisions can be performed algebraically; but, it is often more convenient (in practical factorisation) to find the (numerical) values of L_1, M_1, L_2, M_2, and then perform the divisions (arithmetically).

[When either L_1 or M_1 contains a small divisor p_1, the proper divisor L_1 or M_1 of either L_2 or M_2 is easily found, by determining first (by trial) which of L_2, M_2 contains p_1. Then the Rule is

"The multiple L_1 or M_1 of p_1 is a divisor of the multiple L_2 or M_2 of p_1"...(42).

This process is most useful when $\phi'(m_1) = A_1$, $\phi'(m_2) = A_2$ are *both* Bin-Aurifeuillians.

Ex. $n = 2\omega^2$ gives $N' = (2\omega^2)^{2\omega^2} + 1 = \phi'(2).\phi'(2\omega).\phi'(2\omega^2)$, [Art. 9*b*].

Here $\phi'(2\omega) = \dfrac{F'(2\omega)}{F'(2)}$, $\phi'(2\omega^2) = \dfrac{F'(2\omega^2)}{F'(2\omega)}$, [Art. 9*b*].

Now $F'(2)$, $F'(2\omega)$, $F'(2\omega^2)$ are all Bin-Aurifeuillians so that $\phi'(2\omega)$, $\phi'(2\omega^2)$ are *Quotient Bin-Aurifeuillians*, and may be resolved as above].

Ex. $n = 18$; $N' = (2.3^2)^{2.3^2} = \phi'(2).\phi'(6).\phi'(18)$;

$$\phi'(2) = F'(2) = 18^2 + 1 = (18+1)^2 - 2.18 = (19-6) : (19+6) = 25 : 13; \ = L_1 M_1;$$

$$F'(6) = (18^3)^2 + 1 = (18^3+1)^2 - 2.18^3 = (5833 - 108) : (5833 + 108)$$
$$= 5725 : 5941 = L_2 . M_2;$$

$$\phi'(6) = \frac{F'(6)}{F'(2)} = \frac{5725 : 941}{25 : 13} = 229 : 457;$$

$$F'(18) = (18^9)^2 + 1 = (18^9+1)^2 - 2.18^9 = (18^9+1)^2 - (2^5.3^9)^2$$
$$= 198358660513 : 198359920225 = L_3 . M_3;$$

$$\phi'(18) = \frac{F'(18)}{F(6)} = \frac{L_3 . M_3}{L_2 . M_2} = \frac{L_3}{M_2} \cdot \frac{M_3}{L_2}$$
$$= 33358093 : 37.37.25309;$$

[Here the small factor 5 in L_2 and M_3 shows that L_2 is the divisor of M_3].

PART II. *Arithmetical Factors.*

17. The finding of the arithmetical prime factors (p) of the various A.P.F. of $F(n)$, $F'(n)$ is the most difficult, and most laborious, part of this research. The Theory of Numbers is the guide in this part of the work.

The five following Articles, 18–20*b*, apply equally to all forms of ϕ, ϕ'.

18. *Linear forms of factors* (p). All prime factors (p) of the A.P.F., which occur in this Memoir, must be of the following* *linear forms*,

$$\text{For } \phi(n),\ \phi'(n),\ p = n\varpi + 1;\ \text{ for } \phi(m),\ \phi'(m),\ p = m\varpi + 1 \ldots (43).$$

19. 2^{ic} *forms of factors* (p). The A.P.F. of $F(n)$, &c., can always be (algebraically) expressed in the following forms,

$$n \text{ or } m = 4i+1 \text{ gives } \phi(n \text{ or } m),\ \phi'(n \text{ or } m) = S^2 - mT^2 \ldots\ldots\ldots\ldots (44a),$$

$$n \text{ or } m = 4i-1 \text{ gives } \phi(n \text{ or } m),\ \phi'(n \text{ or } m) = S^2 + mT^2 \ldots\ldots\ldots\ldots (44b),$$

$$n \text{ or } m = 2\mu,\ [\mu = \omega] \text{ gives } \phi'(n \text{ or } m) = a^2 + b^2 = \text{ both } S^2 \mp \mu T^2 \ldots (44c),$$

$$n \text{ or } m = e\mu,\ [e = 2^{\kappa},\ \kappa > 1,\ \mu = \omega], \text{ gives}$$
$$\phi'(n \text{ or } m) = a^2 + b^2 = c^2 \pm 2d^2 = S^2 \mp \mu T^2 = S'^2 \mp 2\mu T'^2 \ldots (44d).$$

[The S, T in the above forms are of course different].

Each of these 2^{ic} forms involves in general a set of linear forms of factors; but the fact of their being representations of A.P.F. of F, F' reduces these to the single type (43) already quoted. These 2^{ic} forms are thus of little use in factorisation, so will not be further treated of.

20. *Quasi-Aurifeuillian Forms.* The *impure* 2^{ic} forms ($P^2 \mp nxy\,Q^2$) treated of in Art. 12–12d form an important help in limiting the linear forms (45) of the arithmetical factors of the various A.P.F. of F, F'.

When the determinant $\pm D = n\,xy$, or $m\,xy$ of those forms, has the form

$$\pm D = n\,xy, \text{ or } m\,xy = \mu s^2 \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (45),$$

then the *impure* 2^{ic} form, in which such A.P.F. would be in general (algebraically) expressible under Art. 12–12d, reduces to a *pure* 2^{ic} form of determinant $\pm D = \mu$, for then

$$\text{The A.P.F.} = P^2 \mp n\,xy\,Q^2, \text{ or } P^2 \mp m\,xy\,Q^2 \text{ becomes } P^2 \mp \mu(sQ)^2 \ldots (46).$$

These functions are here styled *Quasi-Aurifeuillians*, and the conditions (45) which lead to them is styled the *Quasi-Aurifeuillian* condition (from the analogy of the functions treated of in Art. 15). The algebraic values of P, Q may be taken from the Table A of Art. 15.

It will be seen that the *pure* 2^{ic} forms thus produced—depending on particular values of the elements x, y—are

---

* Except that n or m itself is a divisor of $\phi(n \text{ or } m)$ when n or $m = (x \sim y)$, and of $\phi'(n \text{ or } m)$ when n or $m = (x + y)$; but these cases do not occur with the forms (1), (2) of $F(n)$, $F'(n)$ of this Memoir.

additional to, and usually of *different determinant* ($\pm D = \mu$) to, those of Art. 15, which are common to A.P.F. in general (*i.e.* for all values of x, y). Some examples will make this clear.

Ex. Taking $n=15$, the two examples below show for $\phi(15)$ and $\phi'(15)$, when $(x, y) = (\xi^2, 3\eta^2)$, or $(\xi^2, 5\eta^2)$.

Top Line; The three *impure* 2ic forms, common to all x, y.

2nd Line; The Quasi-Aurifeuillian 2ic forms of the above for certain x, y.

3rd Line; The normal *pure* 2ic forms, common to all x, y.

$$\phi(15)\begin{cases} x,\ y,\ 15xy & ;\ P^2+15xyQ^2,\ \ P^2+3xyQ^2\ ,\ \ P^2-5xyQ^2\ ;\ \textit{Impure } 2^{ic}. \\ \xi^2,\ 3\eta^2,\ 5(3\xi\eta)^2 & ;\ P^2+5(sQ)^2\ ,\ \ P^2+(sQ)^2\ ,\ \ P^2-15(sQ)^2\ ;\ \textit{Quas-Aur.} \\ x,\ y, & ;\ t^2-5u^2\ ,\quad .\quad ,\ \ T^2+15U^2\ ;\ \textit{Normal.} \end{cases}$$

$$\phi'(15)\begin{cases} x,\ y,\ 15xy & ;\ P^2-15xyQ^2,\ \ P^2-3xyQ^2\ ,\ \ P^2+5xyQ^2\ ;\ \textit{Impure } 2^{ic}. \\ \xi^2,\ 5\eta^2,\ 3(5\xi\eta)^2 & ;\ P^2-3(sQ)^2\ ,\ \ P^2-15xyQ^2,\ \ P^2+(sQ)^2\ ;\ \textit{Quas-Aur.} \\ x,\ y, & ;\ A^2+3B^2\ ,\ \ T^2+15U^2\ ,\quad .\quad ;\ \textit{Normal.} \end{cases}$$

[Note that the P, Q are of course *different* in each form].

20*a*. *Case of* $\mu = 1$. The important case of the A.P.F. $= P^2 - (sQ^2)$, which gives immediate (algebraic) factorisation, has been fully treated of in Art. 15 under the name *Aurifeuillian*. Referring to that Article it is seen that, under the condition $\mu = 1$ (which is that of Art. 15),

When $\phi(n)$ or $\phi'(n) = P^2 - (sQ)^2$, then $\phi'(n)$ or $\phi(n) = P^2 + (sQ)^2$...(47).

The latter form, conjugate to that of Art. 15, is styled *Ant-Aurifeuillian*. An example will exhibit this property (47) clearly.

Ex. Take $n=27$;

$$F(n) = 27^{27} - 1 = 3^{81} - 1 = \Phi(81);\quad F'(n) = 27^{27} + 1 = 3^{81} + 1 = \Phi'(81).$$

$$\Phi(81) = \phi(1).\phi(3).\phi(9).\phi(27).\phi(81);$$

$$\Phi'(81) = \phi'(1).\phi'(3).\phi'(9).\phi'(27).\phi'(81).$$

Here, except $\phi(1)$ and $\phi'(1)$—

All the $\phi(3)...\phi(81)$ are *Trin-Ant-Aurifeuillians*, (algebraically) expressible in form $P^2 + (sQ)^2$.

All the $\phi'(3)...\phi'(81)$ are *Trin-Aurifeuillians*, (algebraically) expressible in form $P^2 - (sQ)^2 = L.M.$

20*b*. *Quasi-Aurifeuillians, Linear Forms of Factors.*

The *pure* 2ic forms $\{P^2 \mp \mu(sQ)^2\}$ arising from the Quasi-Aurifeuillian condition (45) involve certain definite linear forms of all arithmetical factors (p), usually *different** from that given by (43), which is common to all A.P.F. And, when n is small, these linear forms are simple and few in

* This considerably limits the linear forms possible in such cases.

number. Those for small* values of n or m are tabulated below, up to n or $m \not> 10$.

2^{ic} *form*	a^2+b^2;	c^2+2d^2;	A^2+3B^2;	$t'^2+5u'^2$;	t^2+6u^2;
$p=$	$4\varpi+1$;	$8\varpi+1, 3$;	$6\varpi+1$;	$20\varpi+1, 3, 7, 9$;	$24\varpi+1, 5, 7, 11$;
2^{ic} *form*	t^2+7u^2;	t^2+10u^2;	. ;	$e^2 \sim 2f^2$;	
$p=$	$14\varpi+1, 9, 11$;	$40\varpi+1, 2, 9, 11, 13, 19, 23, 37$;	. ;	$8\varpi \pm 1$;	
2^{ic} *form*	$A'^2-3B'^2$;	t^2-5u^2;	g^2-6h^2;	t^2-7u^2;	t^2-10u^2
$p=$	$12\varpi \mp 1$;	$10\varpi \pm 1$;	$24\varpi \pm 1, 5$;	$28\varpi \mp 1, 3, 9$;	$40\varpi \pm 1, 3, 9, 13$

21. *Residuacity.* The following five Articles (21*a*–*e*) apply only to the A.P.F. of $F(n)$, $F'(n)$ wherein $X=1$; *i.e.* only to the forms $(Y^Y \mp 1)$† of this Paper.

21*a*. 2^{ic} *Residuacity.*

When n or $m = p-1$, then $\phi(n \text{ or } m) = \phi(\tfrac{1}{2}n \text{ or } \tfrac{1}{2}m) \,||\, \phi'(\tfrac{1}{2}n \text{ or } \tfrac{1}{2}m)$ ...(48*a*),

$$\text{then } \phi(\tfrac{1}{2}n \text{ or } \tfrac{1}{2}m) \equiv 0 \pmod{p}, \text{ when } (y/p)_2 = +1 \quad\ldots\ldots\ldots\ldots(48b),$$

$$\text{and } \phi'(\tfrac{1}{2}n \text{ or } \tfrac{1}{2}m) \equiv 0 \pmod{p}, \text{ when } (y/p)_2 = -1 \quad\ldots\ldots\ldots\ldots(48c),$$

Here these cases are at once determinable by the simple laws of 2^{ic} Residuacity.

21*b*. *Residuacity of order* ν.

Let ξ be the *least* exponent satisfying the Congruence

$$y^{\xi} = y^{(p-1)\div\nu} \equiv 1 \pmod{p} \quad\ldots\ldots\ldots\ldots(49),$$

where
$$p = \nu\xi + 1, \quad \xi = (p-1) \div \nu \quad\ldots\ldots\ldots\ldots(49a).$$

Here ξ is styled the *Haupt-Exponent* of y (modulo p), and y is said to be a *Residue of p of order* ν: this last relation is often expressed thus

$$(y/p)_\nu = 1, \text{ which means } y^{(p-1)\div\nu} = 1 \quad\ldots\ldots\ldots\ldots(49b).$$

and here it is clear that

$$m = \xi = (p-1) \div \nu, \text{ with } (y/p)_\nu = 1 \quad\ldots\ldots\ldots\ldots(49c),$$

is the condition that $\phi(m)$ or $\phi'(m) = 0 \pmod{p}$.

21*c*. *Case of* $(Y^Y \mp 1)$. As a result of Art. 18, taking $x = y$,

$$\phi(n) \text{ or } \phi'(n) = \text{M.A.P.F. of } (Y^Y \mp 1) \equiv 0 \pmod{p}$$

requires
$$p = kY + 1 \quad\ldots\ldots\ldots\ldots(50),$$

whence $(-kY)^Y \equiv +1 \pmod{p}$.

Hence
$$Y^Y \mp 1 \equiv 0 \pmod{p}, \text{ if } (-k)^Y \equiv \pm 1 \pmod{p} \quad\ldots\ldots\ldots\ldots(50a).$$

* For the linear forms when n or $m > 10$, see Legendre's *Théorie des Nombres* 3rd Ed., Paris, 1830; t. i., Tab. III. to VII.

† And therefore not to the form $(X^{XY} \mp Y^{XY})$.

When k is *small* (compared to Y) this affords an easy way of testing whether p is a divisor of $\phi(n)$ or $\phi'(n)$.

21*d*. *Residuacity-Rules.* The above test may be written

$$k^{(p-1)\div k} \equiv \pm(-1)^Y \text{ [mod } p\text{]}, \text{ or } (k/p)_k = \pm(-1)^Y \text{.........(50}b\text{)}.$$

Rules are known for determining whether $(z/p)_k = 1$ for the cases of small indices (k), viz.

$$k = 2, 3, 4, 6, 8, 12, 24,$$

but they are dependent on the theory of complex numbers, and are too difficult for ordinary use. They have been reduced to really simple forms for the eight small bases (z)

$$z = 2, 3, 5, 6, 7, 10, 11, 12,$$

but these Rules* are too lengthy to quote here.

21*e*. *Simple Cases.* A simple application is when $k = 2$, 4, 8, 16; the reduced results are shown in the Table below:

p	Y	$Y^Y \equiv +1 \pmod p$	$Y^Y \equiv -1 \pmod p$	
$2Y+1$	.	$p = 8\varpi + 1, 3$;	$p = 8\varpi + 5, 7$	............(51*a*),
$4Y+1$	.	$p = 8\varpi + 1, 5$;	.	............(51*b*),
$8Y+1$	ω	$(2/p)_8 = -1$;	$(2/p)_8 = +1$	............(51*c*),
	ε	$(2/p)_8 = +1$;	$(2/p)_8 = -1$	............(51*d*),
$16Y+1$	ω	$(2/p)_4 = -1$;	$(2/p)_4 = +1$	............(51*e*),
	ε	$(2/p)_4 = +1$;	$(2/p)_4 = -1$	............(51*f*).

To apply these Rules, note that

$$p = 8\varpi + 1 = a^2 + (4\beta)^2 \text{ gives } (2/p)_4 = (\bar{1})^\beta \text{.................(52}a\text{)},$$

$$p = 8\varpi + 1 = (4a \mp 1)^2 + (8\beta)^2 \text{ gives } (2/p)_8 = (\bar{1})^{a+\beta} \text{........(52}b\text{)}.$$

21*f*. *Table of Roots y (mod p).* The short Table B following gives the Results of the above Art. 21—21*e*, *i.e.* the *proper* roots (y) of the Congruences

$$\text{M.A.P.F. of } (y^y - 1) \equiv 0,\ (y^y + 1) \equiv 0 \ (\text{mod } p \ \&\ p^\kappa \not> 1000)\ [y < p \ \&\ p^\kappa] \text{...(53)},$$

omitting however (for shortness' sake) all primes (p) of forms $p = 2y + 1$, $4y + 1$, where y is a prime: as the roots (y) thereof can be at once inferred by the simple Rules of Art. 21*b*,

$$p = 2y + 1 = \begin{cases} 8\varpi + 3, \text{ gives } y^y - 1 \equiv 0 \pmod p, \text{ [}y\text{ prime]}\text{...........(53}a\text{)}, \\ 8\varpi + 7, \text{ gives } y^y + 1 \equiv 0 \pmod p, \text{ [}y\text{ prime]}\text{...........(53}b\text{)}, \end{cases}$$

$$p = 4y + 1 \text{ gives } y^y - 1 \equiv 0, \text{ and } (2y)^{2y} - 1 \equiv 0 \pmod p, \text{ [}y\text{ prime]...(53}c\text{)}.$$

* See two Papers *On the numerical factors of* $(a^n - 1)$ by the late C. E. Bickmore in *Messenger of Maths.*, vol. xxv., 1896, pp. 1–44; and xxvi., 1897, pp. 1–38.

Proper Roots (y) *of* $y^y \equiv \pm 1$ (*mod* p & p^κ), $[y < p]$.

TAB. B.

p	y: +1	y: −1	p	y: +1	y: −1	p	y: +1	y: −1
17	4, 8	.	331	.	.	683	.	.
19	9	.	337	42, 56	21	691	.	.
31	6	.	349	87	58,174	701	175	35,350
37	9	18	353	.	44	709	59, 177	354
41	20	.	367	.	183	727	121	.
43	.	.	373	93	31,186	733	.	.
61	15	30	379	189	63	739	.	.
67	33	.	397	.	.	743	.	371
71	5	35	401	25,100,200	.	751	125	375
73	18	.	409	204	.	757	9, 189	378
79	.	39	419	209	.	761	380	.
89	22	11	421	105	210	769	40,64,192,384	48
97	24, 48	6, 8	431	.	.	787	393	.
101	25	50	433	.	.	809	404	.
103	.	51	439	73	.	811	.	.
109	.	.	443	34, 221	.	821	205	410
113	.	7, 14	449	112, 224	28	823	.	411
127	.	.	457	.	.	827	413	.
131	65	.	461	23, 115	230	829	207	414
137	34, 68	.	487	.	243	853	213	426
139	69	23	491	245	.	857	428	.
151	.	.	499	.	83	859	429	143
157	.	.	521	52, 260	5	877	219	146,438
163	81	27	523	58	.	881	110	55
181	45	30, 90	541	135	270	883	49, 441	.
191	38	95	547	42, 273	.	907	453	.
193	48, 96	12	569	284	.	911	.	7
197	49	98	571	.	95	919	.	.
199	.	11, 99	577	36	.	929	232, 464	58
211	105	.	593	.	37, 74	937	26, 264	.
223	.	.	599	.	299	941	235	470
229	.	.	601	.	.	947	473	43
233	58	29	607	101	303	953	.	.
239	.	119	613	153	306	967	.	483
241	8	.	617	77, 154	.	971	.	.
251	25, 50	.	619	309	.	977	61, 244, 488	.
257	.	4, 8	631	.	.	991	.	495
271	30, 54	135	641	16	32	997	.	83, 166
277	.	.	643	.	107			
281	28,35,70	.	647	.	323			
283	.	.	659	329	47	p^κ	y: +1	y: −1
307	34	51	661	165	33, 330			
311	.	155	673	.	8	29^2	.	14
313	52, 156	.	677	169	338	37^2	.	18

22. *Factorisation-Tables.* At the end of this Paper follow three Tables (I.–III.)—the principal outcome of this Memoir—giving the factorisation into prime factors as completely as practically possible with the means available.

TAB. I., II.; $F(Y) = (Y^Y - 1)$; $F'(Y) = (Y^Y + 1)$; [up to $Y = 50$].

TAB. III.; $F(XY) = (X^{XY} \sim Y^{XY})$; $F'(XY) = (X^{XY} + Y^{XY})$; [up to $XY = 30$].

The following is an Abstract of the degree of completeness of the factorisation attained: the larger numbers are of course very incomplete.

N	*Bases*	*Complete*	*Good Progress*	*Limit*
$F(Y)$;	$Y=$	1 to 16; 18,20,21,22,24,25,30;	27,28,32,34,35,36,40,42,44,45,48;	50
$F'(Y)$;	$Y=$	1 to 16; 18,27;	21,23,24,25,28,30,33,35,36,45;	50
$F(XY)$;	$XY=$	1 to 21; 24, 15.2, 6.5;	28, 10.3	30
$F'(XY)$;	$XY=$	1 to 18; 21;	22, 24, 15.2, 10.3, 6.5	30

The Tables themselves are described in the Articles 22 a–c, following:

22a. *Arrangement of Factors.* Each number F or F' is shown resolved as far as possible into its A.P.F., and those A.P.F. which are Aurifeuillians are (usually) resolved into their twin co-factors (L, M).

Each A.P.F. and each L, M are shown resolved as far as possible into their arithmetical factors.

The A.P.F. are arranged in order of magnitude, the lowest on the left, and the highest (the M.A.P.F.) on the right; and the L precedes the M.

Within each A.P.F., and within each L or M, the arithmetical factors (p and p^κ) are arranged in order of magnitude of the primes (p), the lowest on the left, and the highest on the right.

In incomplete factorisation a blank space is left on the right (of the incomplete A.P.F., L, or M) to admit of the insertion in MS. of new factors.

22b. *Special multiplication symbols* (. | || ; :). These are used to separate various kinds of factors in such a way as to indicate the nature of the factors.

Use of dot (.). This is used between arithmetical factors in the same A.P.F., (but not between the A.P.F. themselves).

A dot on the right of an arithmetical factor, followed by a blank, indicates the existence of other unknown arithmetical factors.

Use of bars (| and ||). These are used between the A.P.F. of $(X^e - Y^e)$, where $e = 2^\mu$, (see Art. 7), thus—

$$X^e - Y^e = (X - Y) \mid (X + Y) \mid (X^2 + Y^2) \mid (X^4 + Y^4) \mid \ldots \parallel (X^{\frac{1}{2}e} + Y^{\frac{1}{2}e}),$$

the double bar (||) being placed just before the M.A.P.F.

Thus the arithmetical factor, or group of factors, between a pair of bars (| ... |) is always an A.P.F. of above form.

Use of semi-colon (;). This is used between A.P.F. not of form $(X^e - Y^e)$. This occurs in both $F(n)$, $F'(n)$ when $n = \omega$, (see Art. 8), and also in $F'(n)$ when $n = e\omega$, (see Art. 9b).

A semi-colon on the extreme right indicates the complete factorisation of the M.A.P.F.

Use of semi-colons (;) *between bars* (| ... |). This occurs in the case of

$F(e\omega)$, which is first resolved into its A.P.F. with respect to the exponent $e = 2^{\mu}$, (see Art. $9a$), thus—

$$X^{e\omega} - Y^{e\omega} = (X^{\omega} - Y^{\omega}) | (X^{\omega} + Y^{\omega}) | (X^{2\omega} + Y^{2\omega}) | (X^{4\omega} + Y^{4\omega}) | \ldots || (X^{\frac{1}{2}e\omega} + Y^{\frac{1}{2}e\omega}).$$

Each of the above A.P.F. of form $(X^{k\omega} = Y^{k\omega})$, where $k = 2^n$, is further resolved (see Art. $9b$) into its A.P.F., which are separated by semi-colons (;), thus taking the form

$$X^{k\omega} - Y^{k\omega} = | \ldots ; \ldots ; \ldots ||.$$

Use of colon (:). This is used between the twin "Aurifeuillian Factors" (L, M) of an Aurifeuillian. These Aurifeuillians occur as complete A.P.F., so that their ends are marked by either bars (|) or semi-colons (;)—[see above].

Use of queries (?). These are used in two ways:—

(1) A query (?) on right of a large arithmetical factor ($>10^7$) indicates that this factor is beyond the powers of the Tables available to resolve or determine primes.

(2) A query on right of the (small) arithmetical factors of an "Aurifeuillian L-Factor" indicates that it is uncertain whether this belongs to the L- or M-factor.

Blank spaces. In the incomplete factorisations blank spaces have been left for the insertion of the (as yet unknown) prime factors in MS.

22c. *Special column-headings.* The entries in four columns on the right, headed *Fac.*, *Aur*, *Lim.*, *In.*, have the following meanings:—

Fac. This column shows the *number* of A.P.F. in $F(Y)$ or $F'(Y)$.

Aur. This column shows the *order* of Aurifeuillians (if any) in $F(Y)$ or $F'(Y)$.

Lim. This column contains symbols (†, ‡, ¶, §) which show—(in case of incomplete factorisation only)—the *limit* to which the search for divisors (p and p^{κ}) has been carried, thus

† to 1000; ‡ to 10000; ¶ to 50000; § to 100000; [or a little further].

[It will be seen that the search has been carried to at least 10000 throughout Tables I., II., and in all but three cases in Tab. III.].

In. This column indicates by "initials (B, C, &c.)—according to the list below—the names (so far as known to the present author) of the original workers who have effected, or have materially contributed to, the various factorisations—

B.	Bickmore, Chas. E.	Lo.	Looff, Dr.
C.	Cunningham, Allan	Lu.	Lucas, Ed.
E.	Euler, L.		

Where no initials are given, the present author is responsible.

Appendix.

In this Appendix is given a short description of the extensive* Tables which were available for the Factorisations of this Paper.

23. *Tables for factors of* $(Y^Y \mp 1)$. It will be seen from Art. $21b$ that the search for factors (p) of $y^m \mp 1 \equiv 0 \bmod p$ is involved in that of finding *proper roots* (y) of the Congruence

$$y^m - 1 \equiv 0 \pmod p, \; [m = \xi = (p-1) \div \nu] \ldots\ldots\ldots\ldots\ldots (51).$$

* These Tables have been over 20 years under preparation.

The search is therefore dependent chiefly on Tables of the solutions, *i.e. proper roots* (y) of that Congruence. The Tables available are described in the Art. 23*a*–*d* following.

23*a*. *Reuschle's Tables.* These Tables* give the complete† set of roots ($y < \frac{1}{2}p$) of the Congruence

$$y^m - 1 \equiv 0 \pmod{p \not> 1000},$$

for the following values of m:—

m = every *odd* prime, and prime power < 100,
every *odd* composite up to 69 (except 65),
every power of 2 up to $2^7 = 128$.
every multiple of 4 up to 100 (except 88, 92) and 120.

The roots (y') of $y'^m + 1 \equiv 0$ are not especially mentioned, but appear as follows:—

The roots ($y' < \frac{1}{2}p$) of $y'^m + 1 \equiv 0$, with $m = \omega$, appear as *negative roots* ($-y$) of $y^m - 1 \equiv 0$, with $m = \omega$

The roots ($y' < \frac{1}{2}p$) of $y'^m + 1 \equiv 0$, with $m = \epsilon$, appear as roots (y) of $y^{2m} - 1 \equiv 0$.

It will be seen that these tables give a very extended range of the index (m); but the range of the modulus (p) is so restricted ($p \not> 1000$) that their use in factorisation is very limited.

23*b*. *The author's Tables.* The author has had extensive Tables of this sort‡ compiled, giving the complete set of *proper* roots y, y':—

When $m = \omega$; of $y^m - 1 \equiv 0$, and $y'^m + 1 \equiv 0 \pmod{p \text{ and } p^\kappa}$.
When $m = \epsilon$; of $y'^m + 1 \equiv 0 \pmod{p \text{ and } p^\kappa}$

for the values of m stated below, and up to the limits of p and p^κ stated:—

$m = 2, 3, 4, 6, 8, 12$	5, 7, 9,	10, 11, 13, 14, 15;
p and $p^\kappa \not> 100000$	60000	50000 (or a little over).

23*c*. *Creak's Tables.* Mr. T. G. Creak has compiled Tables‡§ of the same sort as the above for the values m, stated below, and within the limits of p and p^κ stated:—

$m = 16$ to 50, 52, 54, 56, 63, 64, 72, 75 (mod p and $p^\kappa > 10^3$ up to 10^4).

23*d*. *Small bases* $y \not> 12$. Besides the above, the author has—in conjunction with Mr. H. J. Woodall, A.R.C.Sc., compiled Tables giving the Haupt-Exponents (ξ) and Max.-Residue Indices (ν) of the following Bases (y):—

$y = 2$,‖ for all primes and prime-powers $\not> 100000$.
$y = 3, 5, 6, 7, 11, 12$,¶ for all primes and prime-powers $\not> 15000$.
$y = 3, 5, 6, 7, 11, 12$,** for all $\xi = \omega \not> 105$ and $= \epsilon \not> 210$, for all primes and prime-powers $\not> 50000$.

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* *Tafeln complexer Primzahlen*, by Dr. C. G Reuschle, Berlin, 1875.

† Some errata have been found in this part of these Tables. A list of these will be published hereafter.

‡ These Tables are now in course of publication.

§ These Tables Mr. Creak has kindly placed at the author's disposal.

‖ In five papers on "Haupt-Exponents of 2" in the *Quarterly Journal of Math.*, vol. xxxvii, xlii., xliv., xlv.; 1905–1914.

¶ In course of publication.

** At present only in MS.

24. *Tables for factors of* $(X^{XY} \mp Y^{XY})$.

The Tables available for this purpose are described in Art. 24*a* below. They differ in use from the Tables described in Art. 22*a*–*d* for factorising the simpler forms $(Y^Y \mp 1)$ in that the particular A.P.F. of $F(n)$ or $F'(n)$ to which the divisors formed belong cannot always be identified from the Tables themselves.

These Tables are also not nearly so extensive as the previous ones, so that the factorisation of $(X^{XY} \mp Y^{XY})$ cannot be carried to such high limits as in the simpler case.

24*a*. *Canon Arithmeticus.* This Canon* gives two kinds of Tables for every prime (p) and prime-power (p^κ) as moduli up to p and $p^\kappa \not> 1000$. One Table gives the *Least + Residue* (R) of all the powers g^ρ of the base g, up to the limit $\rho < p$ or p^κ, (*i.e.*, it gives R to Argument ρ). The other Table gives ρ to Argument R, with same limits.

Here g is in every case some *primitive root*† of the modulus (p or p^κ).

Use of the Table. The right-hand Table gives the powers (g^α, g^β), such that

$$g^\alpha \equiv x, \text{ and } g^\beta \equiv y \pmod{p \text{ or } p^\kappa}.$$

Hence $\quad x^m \mp y^m \equiv g^{m\alpha} \mp g^{m\beta} \equiv g^{m\beta}(g^{m\alpha - m\beta} \mp 1)$, [mod p or p^κ].

Hence $\quad x^m - y^m \equiv 0 \pmod{p}$, if $m(\alpha - \beta) \equiv 0$ [mod $(p-1)$],

$$x^m - y^m \equiv 0 \pmod{p^\kappa}, \text{ if } m(\alpha - \beta) \equiv 0 \;[\text{mod } \tau],$$

$$x^m + y^m \equiv 0 \pmod{p}, \text{ if } m(\alpha - \beta) \equiv 0 \;[\text{mod } \tfrac{1}{2}(p-1)],$$
$$\text{but not } \equiv 0 \;[\text{mod } (p-1)],$$

$$x^m + y^m \equiv 0 \pmod{p^\kappa}, \text{ if } m(\alpha - \beta) \equiv 0 \;[\text{mod } \tfrac{1}{2}\tau],$$
$$\text{but not } \equiv 0 \;[\text{mod } \tau],$$

where $\quad \tau = (p-1) . p^{\kappa - 1}$.

This Table suffices for finding *all* the divisors p and $p^\kappa \not> 1000$ for *all bases* (x, y) whatever, because the Base (g) of each Table is always a *primitive root* of the modulus (p or p^κ).

[The use (in factorisation) is very limited on account of the restricted limit of the moduli (p and $p^\kappa \not> 1000$)].

24*b*. *Binary Canon.* This Canon‡ is *quite similar* to the *Canon Arithmeticus* (Art. 24*a*), and has the same scope. It differs only in that the Base 2 is used in every Table throughout (instead of a primitive root, g).

It can be used in precisely the same way as described in Art. 24*a* : but its use is of course limited to bases (x, y) such that real values (α, β) exist giving $2^\alpha \equiv x$, $2^\beta \equiv y \pmod{p \text{ or } p^\kappa}$.

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* *Canon Arithmeticus*, by C. G. J. Jacobi, Berlin, 1839. This Canon has unfortunately many Errata ; the Appendix contains five 4to pages of these. The present author has found a few more : a list of these will be given hereafter.

† The primitive roots (g) selected are frequently so large as to be very inconvenient for numerical calculations, *e.g.* with $p = 997$, the chosen $g = 656$. Fortunately this is of no importance when (as is usual) only the Residues of g^ρ are required.

‡ *Binary Canon*, London, 1900, by the present author, prepared for the British Association.

24*c*. *Other Canons.* The author has had Tables* prepared giving (at sight) the Least Residues (R, R'), both + and −, of the powers (z^ρ) of the small Bases (z) named below on division by all primes (p) and prime-powers (p^κ) up to the limits of ρ, p, named below:—

Base z =	2†	2†	3, 5, 7, 10†, 11.
Powers of z; $\rho \not>$	100	36	30
Moduli; p & $p^\kappa \not>$	10000	12000	10000

These Tables suffice for finding divisors (p and p^κ) of $(x^m \mp y^m)$ where x, y are any of the above-named Bases (z), or small powers thereof, up to the limits of $m = \rho$, and p, p^κ named.

24*d*. *Special Congruence Tables.* Two sets of Tables of the same kind and scope were‡ available, connecting the auxiliary Bases 2 or 10 with each of the Bases $y = 3, 5, 7, 11$ by the Congruences:—

$$\text{i.§} \quad 2^{x_0} \equiv \pm y^{a_0}, \text{ and } 2^{x_0'} . y^{a_0} \equiv \pm 1 \pmod{p \text{ or } p^\kappa \not> 10^4}.$$

$$\text{ii.§} \quad 10^{x_0} \equiv \pm y^{a_0}, \text{ and } 10^{x_0'} . y^{a_0} \equiv \pm 1 \pmod{p \text{ or } p^\kappa \not> 10^4}.$$

The Tables give (at sight) the solutions (x_0, x_0', a_0 and the ± sign) of the above Congruences for each of the small Bases $y = 3, 5, 7, 11$.

In both Tables i., ii.

a_0 denotes the absolute *minimum* exponent possible for the Base y.

x_0, x_0' mean the least exponent of the auxiliary Base y going with the exponent a_0 of y.

From these Tables may be formed, by aid of the Haupt-Exponents (ξ_2, ξ_{10}) of the Base 2, 10, all possible Congruences connecting the auxiliary Bases 2, 10 with the other Bases $y = 3, 5, 7, 11$:—

$$\text{i. } 2^x \equiv \pm y^a,\ 2^x.y^a \equiv \pm 1;\quad 10^x \equiv \pm y^a,\ 10^x.y^a \equiv \pm 1 \pmod{p \ \& \ p^\kappa > 10^4}.$$

In all such Congruences the tabular exponent a_0 is a necessary factor of the exponents a possible to y.

Hence these Tables are suitable for finding factors (p and $p^\kappa > 10^4$) *directly* of numbers of following forms, [$y = 3, 5, 7, 11$]:—

$$\text{i. } (2^x \mp y^a),\ (2^x.y^a \mp 1);\quad \text{ii. } (10^x \mp y^a),\ (10^x.y^a \mp 1).$$

They may also be used—with some additional trouble—for finding factors (p and $p^\kappa \not> 10^4$) of the forms

$$(v^a \mp w^b),\quad (v^a.w^b \mp 1),$$

where v, w are any of the Bases 2, 3, 5, 7, 11, or any of their powers, or any products thereof.

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* All at present only in MS.

† The Tables of Bases 2 and 10 were prepared by the author and Mr. H. J. Woodall, of Stockport, jointly (but independently).

‡ These are now in course of publication.

§ The Tables i. were prepared by Mr. H. J. Woodall and the present author conjointly (but independently). The Tables ii. were prepared by Mr. H. J. Woodall and Mr. T. G. Creak conjointly (but independently).

Factorisation Table of $F(Y)=(Y^Y-1)$. TAB. I.

<table>
<tr><th>Y</th><th>F(Y) = Y^Y − 1</th><th>Fac</th><th>Aur</th><th>Lm</th><th>In</th></tr>
<tr><td>1</td><td>0;</td><td>1</td><td>.</td><td></td><td></td></tr>
<tr><td>2</td><td>1|3;</td><td>2</td><td>.</td><td></td><td></td></tr>
<tr><td>3</td><td>2; 13;</td><td>2</td><td>.</td><td></td><td></td></tr>
<tr><td>4</td><td>3|5||17;</td><td>3</td><td>.</td><td></td><td></td></tr>
<tr><td>5</td><td>4; 11:71;</td><td>2</td><td>5</td><td></td><td></td></tr>
<tr><td>6</td><td>5; 43||7; 31;</td><td>4</td><td>.</td><td></td><td></td></tr>
<tr><td>7</td><td>2.3; 29.4733;</td><td>2</td><td>.</td><td></td><td>E</td></tr>
<tr><td>8</td><td>7|3; 3|5; 13||17; 241;</td><td>8</td><td>.</td><td></td><td>E</td></tr>
<tr><td>9</td><td>2; 13; 757||4; 1:7; 19:37;</td><td>6</td><td>3</td><td></td><td>E</td></tr>
<tr><td>10</td><td>9; 41.271||11; 9091;</td><td>4</td><td>.</td><td></td><td>Lo</td></tr>
<tr><td>11</td><td>2.5; 15797.1806113;</td><td>2</td><td>.</td><td></td><td>B</td></tr>
<tr><td>12</td><td>11; 157|13; 17:19||5.29; 20593;</td><td>6</td><td>3</td><td></td><td>B</td></tr>
<tr><td>13</td><td>4.3; 1803647:53.264031;</td><td>2</td><td>13</td><td></td><td>C</td></tr>
<tr><td>14</td><td>13;8108731||3.5; 70275567;</td><td>4</td><td>.</td><td></td><td>Lu</td></tr>
<tr><td>15</td><td>2.7; 241; 11.4931; 61.39225301;</td><td>4</td><td>.</td><td></td><td>C</td></tr>
<tr><td>16</td><td>1|3|5|17|257|65537||641.6700417;</td><td>7</td><td>.</td><td></td><td>E</td></tr>
<tr><td>17</td><td>16; 2699538733?:19152352117?</td><td>2</td><td>17</td><td>‡</td><td></td></tr>
<tr><td>18</td><td>17;343; 991.34327||19; 307; 73.465841;</td><td>6</td><td>.</td><td></td><td>C</td></tr>
<tr><td>19</td><td>2.9;</td><td>2</td><td>.</td><td>‡</td><td></td></tr>
<tr><td>20</td><td>19; 251:11.61|3.7; 152381||401; 41.2801.222361;</td><td>6</td><td>5</td><td></td><td>Lu</td></tr>
<tr><td>21</td><td>4.5; 463; 43.631.3319; 4789.6427:227633407;</td><td>4</td><td>21</td><td></td><td>C</td></tr>
<tr><td>22</td><td>3.7; 67.353.1176469537||23; 89.285451051007;</td><td>4</td><td>.</td><td></td><td>Lu</td></tr>
<tr><td>23</td><td>2.11; 461.1289.</td><td>2</td><td>.</td><td>‡</td><td></td></tr>
<tr><td>24</td><td>23; 601|25; 7.79|577; 349:13.73||331777; 97.1134793633;</td><td>8</td><td>6</td><td></td><td>Lu</td></tr>
<tr><td>25</td><td>4; 11.71; 9384251:101.251.401||2.3; 521; 1901.50150933101;</td><td>6</td><td>5</td><td></td><td>C</td></tr>
<tr><td>26</td><td>25; ||27; 937.6449.38299.397073;</td><td>4</td><td>.</td><td>¶</td><td>C</td></tr>
<tr><td>27</td><td>2; 13; 757; 109.433.8209; 3889.</td><td>5</td><td>.</td><td>¶</td><td>BC</td></tr>
<tr><td>28</td><td>27; 113.4422461|29; 13007:35771||5.137; 281.</td><td>6</td><td>.</td><td>¶</td><td>LuC</td></tr>
<tr><td>29</td><td>4.7; 59.?</td><td>2</td><td>29</td><td>‡</td><td></td></tr>
<tr><td>30</td><td>29;49.19;837931;12211.519411161||31;13 67;11.71261;271 4831.517831;</td><td>8</td><td>.</td><td></td><td>Lu</td></tr>
<tr><td>31</td><td>2.3.5;</td><td>2</td><td>.</td><td>‡</td><td></td></tr>
<tr><td>32</td><td>31|3; 11|5; 5.41|17; 61681|257; 4275255361||65537;</td><td>12</td><td>.</td><td>§</td><td>ELu</td></tr>
<tr><td>33</td><td>32; 1123; 2113. ; 67.</td><td>4</td><td>33</td><td>¶</td><td></td></tr>
<tr><td>34</td><td>3.11; 103.137.
||5.7; 307.443.1531.112643.28051.4708729;</td><td>4</td><td>.</td><td>‡</td><td>C</td></tr>
<tr><td>35</td><td>2.17; 31.49831; 43.440077727; 281.</td><td>4</td><td>.</td><td>‡</td><td>C</td></tr>
<tr><td>36</td><td>5; 43; 19.2467|7; 31; 46441|37; 13.97; 73.541:55117
||1297; 1678321; 577.3313.24787501869 61?</td><td>12</td><td>6</td><td>¶</td><td>C</td></tr>
<tr><td>37</td><td>4.9; 149.1999.7993.?</td><td>2</td><td>37</td><td>‡</td><td></td></tr>
<tr><td>38</td><td>37; ||3.13; 191.</td><td>4</td><td>.</td><td>‡</td><td></td></tr>
<tr><td>39</td><td>2.19; 7.223; 53.131.157. ; 3121.</td><td>4</td><td>.</td><td>‡</td><td></td></tr>
<tr><td>40</td><td>3.13; 2625641|41; 121.20641|1601; 281.5501:241.17581||
||769.3329;</td><td>8</td><td>10</td><td>¶</td><td>C</td></tr>
<tr><td>41</td><td>8.5; 83?</td><td>2</td><td>11</td><td>‡</td><td></td></tr>
<tr><td>42</td><td>41; 13.139; 3851.1460117; 1009. ||
||43; 1723; 29.337.548591; 547.19489.</td><td>8</td><td>.</td><td>‡</td><td>C</td></tr>
<tr><td>43</td><td>2.3.7; 173.6709.</td><td>2</td><td>.</td><td>‡</td><td></td></tr>
<tr><td>44</td><td>43; 6337. |9.5; 23.4316489:89.991.3037||
||13.149;</td><td>6</td><td>1.</td><td>¶</td><td>C</td></tr>
<tr><td>45</td><td>4.11; 19.109; 1471:2851; 10009.829639; 2891101:31.183451;
181.</td><td>6</td><td>5</td><td>‡</td><td>C</td></tr>
<tr><td>46</td><td>9.5;
||47;</td><td>4</td><td>.</td><td>‡</td><td></td></tr>
<tr><td>47</td><td>2.23; 1693.?</td><td>2</td><td>.</td><td>‡</td><td></td></tr>
<tr><td>48</td><td>47; 13.181|49; 37:61|5.461; 5306113|5308417; 8929.3155927939||
||17.113. ; 97.193.</td><td>10</td><td>3</td><td>¶</td><td>C</td></tr>
<tr><td>49</td><td>2.3; 29.4733; 3529. ||
8; 113:911; 197.883. :3823.</td><td>6</td><td>7</td><td>‡</td><td>C</td></tr>
<tr><td>50</td><td>49; 6377551; 151. ||
||3.17; 11.557041; 251.</td><td>6</td><td>.</td><td>‡</td><td>C</td></tr>
</table>

Factorisation Table of $F'(Y) = (Y^Y + 1)$. TAB. II.

Y	$F'(Y) = Y^Y + 1$	*Fac*	*Aur*	*Lm*	*In*
1	2;	1	.		
2	1:5;	2	2		
3	4; 1:7;	3	3		
4	257;	1	.		
5	2.3; 521;	2	.		
6	37; 13:97;	2	6		
7	8; 113:911;	2	7		
8	257; 97 673;	2	.		
9	2.5; 73; 530713;	3	.		C
10	101; 3541:27961;	2	10		Lo
11	3.4; 23.89.199:58367;	2	11		B
12	89.233; 193.2227777;	2	.		B
13	2.7; 13417.20333.79301;	2	.		C
14	197; 29 29.3361:113.176597;	2	14		Lu
15	16; 211; 31.1531; 19231:142111;	4	15		C
16	274177.67280421310721;	1	.		Lu
17	2.9;	2	.	‡	
18	13:25; 229:457; 33388093:37.37.25309;	6	2		B C
19	4.5; 113631466919? :870542161121?	2	19	‡	
20	160001;	2	.	¶	
21	2.11; 421; 81867661; 337.	4	.	¶	B C
22	5.97;	2	22	¶	
23	8.3; 47.139.1013.52626071:2498077661567473?	3	23	‡	
24	17.2801.2311681; 33409.	2	.	¶	
25	2.13; 41.9161;	3	.	¶	
26	677; 53.	2	26	¶	
27	4; 1:7; 19:37; 19441:19927; 163.208657.224209:1297.5879415781;	5	3		C
28	614657; 449.23633.	2	.	¶	
29	2.3.5; 233.	2	.	‡	
30	17.53; 809101; ; 61.181.21872881:1784464680181?	4	30	¶	C
31	3^2, 373.1613.? :	2	31	‡	
32	641.6700417;	2	.	δ	E
33	2.17; 7.151; 23.1871.34544013769? ; 661.	4	.	¶	
34	13.89;	2	34	‡	
35	4.9; 11.132631; 29.5209.11831; 71.701.?	4	35	‡	
36	17.98801; 5953 473896897;	3	.	¶	C
37	2.19; 593.	2	.	‡	
38	5.17.17;	4	38	‡	
39	8.5; 1483; ; 79.? :	5	39	‡	
40	17.17.113.337.641.929;	2	.	‡	C
41	2.3.7;	2	.	‡	
42	5.353; 673.4621; ;	4	42	‡	C
43	4.11; 947.1291.? :	2	43	‡	
44	41.113.809; 353.9857.	2	.	‡	
45	2.23; 7.283; 41.97841; 7309.1136089; 61 ;	6	.	‡	C
46	29.73; 1013.? :	2	46	‡	
47	16.3; 659.?	2	47	‡	
48	; 769.	2	.	‡	
49	2.25; 13564461457; 16073.	3	.	‡	C
50	41.61; 5122541:7622561; 101.?	5	2	‡	C

Factorisation Table of $(X^{XY} \mp Y^{XY})$. TAB. III.

<table>
<tr><th>$(X^{XY} \sim Y^{XY})$</th><th>Fac</th><th>Aur</th><th>Lim</th><th>X, Y</th><th>$(X^{XY} + Y^{XY})$</th><th>Fac</th><th>Aur</th><th>Lim</th></tr>
<tr><td>1; 19||5; 7;</td><td>4</td><td>.</td><td></td><td>3, 2</td><td>13; 1:61;</td><td>2</td><td>6</td><td></td></tr>
<tr><td>3; 1031||7; 11.41;</td><td>4</td><td>.</td><td></td><td>5, 2</td><td>29; 61:5521;</td><td>2</td><td>10</td><td></td></tr>
<tr><td>1; 37|7; 13||25; 193;</td><td>6</td><td>.</td><td></td><td>4, 3</td><td>337; 51361;</td><td>2</td><td>.</td><td></td></tr>
<tr><td>5; 164683||9; 71.1289;</td><td>4</td><td>.</td><td></td><td>7, 2</td><td>53; 29.617:113.6329;</td><td>2</td><td>14</td><td></td></tr>
<tr><td>2; 49; 11.131; 61.3541;</td><td>4</td><td>.</td><td></td><td>5, 3</td><td>8; 19; 421; 31:15391;</td><td>4</td><td>15</td><td></td></tr>
<tr><td>7; 103; 127.4231||11; 67; 19.73.379;</td><td>6</td><td>.</td><td></td><td>9, 2</td><td>5.17; 37:13.13; 397.1153:616933;</td><td>3</td><td>2</td><td></td></tr>
<tr><td>1; 11:191|9; 461||41; 263761;</td><td>3</td><td>5</td><td></td><td>5, 4</td><td>881; 109497097921?</td><td>2</td><td>.</td><td>‡</td></tr>
<tr><td>4; 79; 205339; 4159:43.127.379;</td><td>4</td><td>21</td><td></td><td>7, 3</td><td>2 5; 37; 71.1163; 3529.5180449;</td><td>4</td><td>.</td><td></td></tr>
<tr><td>9; 31701296507?||13; 23.89.10721569;</td><td>4</td><td>.</td><td>‡</td><td>11, 2</td><td>125; 353.8764933:210477537689?</td><td>2</td><td>22</td><td>‡</td></tr>
<tr><td>5:97|11; 49|73; 13:277||4177; 577.28513;</td><td>8</td><td>6</td><td></td><td>8, 3</td><td>17.113.8737; 28136494444320I?</td><td>2</td><td>.</td><td></td></tr>
<tr><td>11; 79. ||3.5; 313.</td><td>4</td><td>.</td><td>†</td><td>13, 2</td><td>173; 53.? :</td><td>2</td><td>26</td><td>†</td></tr>
<tr><td>3; 113.2381|11; 29 2633||5.13; 449.23247953;</td><td>6</td><td>.</td><td></td><td>7, 4</td><td>2657; 3361.</td><td>2</td><td>.</td><td>‡</td></tr>
<tr><td>13; 7.37; 58411; 2226543211?||17; 199; 11.31.131; 3001.965551;</td><td>8</td><td>.</td><td>‡</td><td>15, 2</td><td>229; 49741; 61.1801.22921;</td><td>4</td><td>30</td><td>†</td></tr>
<tr><td>7; 139; 14251; 72117691||13; 79; 11.701; 31.271.15031;</td><td>8</td><td>.</td><td></td><td>10, 3</td><td>109; 9181; 91743661; 61.61.1381:2119600921?</td><td>4</td><td>30</td><td>‡</td></tr>
<tr><td>1; 7.13; 4651; 10388II||11; 31; 991; 1481671;</td><td>8</td><td>.</td><td></td><td>6, 5</td><td>61; 1021; 181.6361; 7561:409472221?</td><td>4</td><td>30</td><td>†</td></tr>
</table>

Appendix II.

Errata in Ed. Lucas's Tables of $(Y^2 - nz Z^2)$.

1°. *Comptes Rendus de l'Association Française pour l'Avancement des Sciences*, Paris, 1878, pp. 164—173. *Sur les Formules de Cauchy et de Lejeune-Dirichlet.* By Ed. Lucas.

page 168. Col. of "Coefficients de Y_1":

Line of $n=29$. *For* $33+15$, *Read* $33+13+15$.

Line of $n=33$. *For* $-19]$, *Read* $+19]$.

Line of $n=41$. *For* -57, *Read* $+67$.

2°. Separate Reprint of above Paper, Paris, 1878:

page 5, line 13. *For* z_1, *Read* Z_1.

page 5. Col. of "Coefficients de Y_1",

Line of $n=29$. *For* $33+15$, *Read* $+33+13+15$.

3°. *Sur la Série récurrente de Fermat*, by Ed. Lucas, Rome, 1879.

page 6. Table of "Formules de Mm. Le Lasseur, &c."

In the formulæ for Y:

Line of $p=22$. *For* $+x^5y^5$, *Read* $+11x^5y^5$.

Line of $p=29$. *For* $+15x^{11}y^3$, *Read* $+13x^{11}y^3$.

Line of $p=33$. *For* $-19x^5y^5+$, *Read* $-59x^5y^5-$.

The Tables in the above Memoirs are not identical: they differ only in the signs ($\pm$) of the coefficients A_r, B_r when $n=4i+3$ for all odd values of r.

The signs in the Table of Memoir 1° apply directly to $\phi(n)$: those in Paper 3° apply directly to $\phi'(n)$, when $n=4i+3$, this being the factorisable Aurifeuillian form: these signs are adopted in Tab. A of the present Memoir.

NOTE ON CLASS RELATION FORMULÆ.

By *L. J. Mordell*, Birkbeck College, London.

Let $F(m)$ be the number of uneven classes, $G(m)$ the whole number of classes of forms of determinant $-m$, the classes $(1, 0, 1)$, $(2, 1, 2)$ and their derived classes being counted as $\frac{1}{2}$ and $\frac{1}{3}$ respectively; to $F(0)$ we attribute the value 0, to $G(0)$ the value $-\frac{1}{12}$. Let a be any divisor of m which is $\leq \sqrt{m}$ and of the same parity as its conjugate divisor d. In any summation involving a, we take $\frac{1}{2}a$ instead of a when $a = \sqrt{m}$. Further, call any divisor of m, b or c according as its conjugate divisor is odd or even.

Put
$$Q = \sum_{n=0}^{\infty} F(n)\, q^n,$$

$$R = \sum_{n=0}^{\infty} [4F(n) - 3G(n)]\, q^n,$$

where as usual $q = e^{\iota\pi\omega}$. Then

$$8Q\theta_{00}(x) = -\sum_{n=\pm\infty} 2nq^{n^2} e^{2n\pi\iota x}\left(\frac{1+q^{2n}e^{2\iota\pi x}}{1-q^{2n}e^{2\iota\pi x}}\right) + \frac{1}{\pi}\frac{\theta_{10}(x)}{\theta_{11}(x)}\theta'_{01}(x)\theta_{00} \ldots (A),$$

$$4R\theta_{00}(x) = \sum_{n=\pm\infty} 2nq^{n^2} e^{2n\pi\iota x}\left(\frac{1+q^{2n}e^{2\iota\pi x}}{1-q^{2n}e^{2\iota\pi x}}\right) - \frac{1}{\pi}\frac{\theta_{01}(x)}{\theta_{11}(x)}\theta'_{10}(x)\theta_{00} \ldots (B),$$

where $n = 0$ is omitted from the right-hand summation,

and
$$\theta'(x) = \frac{d\,\theta(x)}{dx}, \quad \theta_{00} = \theta_{00}(0, \omega).$$

We also have more or less similar equations when $\theta_{00}(x, \omega)$ on the left-hand side is replaced by many other functions, *e.g.* $\theta_{00}(x, m\omega)$, or again when Q and R are replaced by series in which the coefficient of q^n is equal to the class number of particular kinds of quadratic forms of determinant $-n$, *e.g.* taking amongst those reduced forms whose third coefficient is odd the excess of the number of those whose

first coefficient is odd over the number whose first coefficient is even.

The derivation of formulæ of this kind is extremely simple, and will form the subject of a paper entitled "Class relation formulæ," which I hope will appear in due course of time in the *Quarterly Journal of Mathematics*. I may notice, however, a few obvious applications of the formulæ A and B.

Putting $x = \frac{1}{2}$ in (A), we find

$$8Q\theta_{01} = 4\sum_{n=0}^{\infty}(-1)^{n+1} nq^{n^2}\left(\frac{1-q^{2n}}{1+q^{2n}}\right) \text{............}(1'),$$

$$= 8\sum_{n=0}^{\infty}\sum_{t=0}^{\infty}(-1)^{n+t+1} nq^{n^2+2nt},$$

where if t is zero, we write $\frac{1}{2}n$ for the coefficient n. Equating coefficients of q^m, we have

$$F(m) - 2F(m-1^2) + 2F(m-2^2) - \ldots = \Sigma a(-1)^{\frac{1}{2}(a+d)+1} \ldots (1).$$

But, putting $x = 0$, we have

$$8Q\theta_{00} = -4\sum_{n=1}^{\infty} nq^{n^2}\left(\frac{1+q^{2n}}{1-q^{2n}}\right) + \frac{1}{\pi}\frac{\theta_{10}\theta_{00}\theta_{01}''}{\theta_{11}'}.$$

The second term on the right-hand side can be written as

$$\frac{1}{\pi^2}\frac{\theta_{01}''}{\theta_{01}} = \frac{1}{\pi^2}\frac{d}{dx}\left[\frac{\theta_{01}'(x)}{\theta_{01}(x)}\right], \text{ when } x = 0,$$

and this is found to be

$$8\sum_{n=1}^{\infty}\frac{q^{2n-1}}{(1-q^{2n-1})^2},$$

so that equating coefficients of q^m, we find

$$F(m) + 2F(m-1^2) + 2F(m-2^2) + \ldots = -\Sigma a + \Sigma b \ldots (2).$$

Putting now $x = \frac{1}{2}$ in (B),

$$4R\theta_{01} = 4\sum_{n=0}^{\infty}(-1)^n nq^{n^2}\left(\frac{1-q^{2n}}{1+q^{2n}}\right) + \frac{1}{\pi}\frac{\theta_{00}\theta_{11}'\theta_{00}}{\theta_{10}}$$

$$= 4\sum_{n=0}^{\infty}(-1)^n nq^{n^2}\left(\frac{1-q^{2n}}{1+q^{2n}}\right) + \theta_{00}^3\theta_{01},$$

and adding this to equation (1′), we have

$$4R + 8Q = \theta_{00}^{\ 3}$$

$$= \sum_{n=0}^{\infty} [24F(n) - 12G(n)]\, q^n \text{ ....(3).}$$

Putting $x = 0$ gives

$$4R\,\theta_{00} = \sum_{n=1}^{\infty} 4nq^{n^2}\left(\frac{1+q^{2n}}{1-q^{2n}}\right) - \frac{1}{\pi}\,\frac{\theta_{01}\theta_{10}{''}\theta_{00}}{\theta_{11}{'}}.$$

But $-\dfrac{1}{\pi}\,\dfrac{\theta_{01}\theta_{00}\theta_{10}{''}}{\theta_{11}{'}} = -\dfrac{1}{\pi^2}\,\dfrac{\theta_{10}{''}}{\theta_{10}} = -\dfrac{1}{\pi^2}\,\dfrac{d}{dx}\left[\dfrac{\theta_{10}{'}(x)}{\theta_{10}\ (x)}\right]$ when $x = 0$,

$$= 1 + 8\sum_{n=1}^{\infty}\frac{q^{2n}}{(1+q^{2n})^2},$$

so that equating coefficients of q^m,

$$8\sum_r F(m - r^2) - 6\sum_r G(m - r^2) = 4\,\Sigma a + 4\,\Sigma(-1)^{c+1}c,$$

but $\quad 8\sum_r F(m - r^2) \qquad = -8\,\Sigma a + 8\,\Sigma b,$

where the summation for r extends to all positive, negative, and zero values for which $m - r^2$ is not negative, as is customary in such summations; so that by subtraction

$$3\left[G(m) + 2G(m - 1^2) + 2G(m - 2^2) + \ldots\right]$$

$$= -6\,\Sigma a + 4\,\Sigma b + 2\,\Sigma(-1)^c c \text{......(4).}$$

We can find other formulæ by putting $x = \frac{1}{2}\omega$, $\frac{1}{2}(1+\omega)$ in equations A, B, though we have to differentiate these formulæ before we write $x = \frac{1}{2}(1+\omega)$.

Equations 1, 2 were given by Kronecker in a slightly different form; as was equation (4), only in the particular case, however, when m is odd.* Equation (3) seems worthy of a moment's consideration, for it gives us instantly the fact that the number of solutions of $x^2 + y^2 + z^2 = n$ is $24F(n) - 12G(n)$. And multiplying the equation throughout by θ_{00}, and making use of equations (2, 4), we find that the number of solutions of $x^2 + y^2 + z^2 + t^2 = m$ is

$$24(-\Sigma a + \Sigma b) - 4\{-6\Sigma a + 4\Sigma b + \Sigma 2(-1)^c c\}$$

or $\qquad 8\{\Sigma b - \Sigma(-1)^c c\}.$

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* See H. J. S. Smith, *Report on Theory of Numbers*, *Collected Works*, vol. i., page 343; and page 324 for equation 3.

But more than this, it contains implicitly the following well-known equations, due to Kronecker,*

$$4\sum_{n=0}^{\infty} F(4n+1)\,q^{\frac{1}{4}(4n+1)} = \theta_{10}\theta^2_{00},$$

$$4\sum_{n=0}^{\infty} F(4n+2)\,q^{\frac{1}{4}(4n+2)} = \theta^2_{10}\theta_{00},$$

$$8\sum_{n=0}^{\infty} F(8n+3)\,q^{\frac{1}{4}(8n+3)} = \theta^3_{10}.$$

For remembering that

$$F(n) = G(n) \quad \text{if} \quad n \equiv 1,\ 2 \bmod 4,$$

$$2F(n) = G(n) \quad ,, \quad n \equiv 7 \bmod 8,$$

$$4F(n) = 3G(n) \quad ,, \quad n \equiv 3 \bmod 8,$$

and putting successively ιq, $\iota^2 q$, $\iota^3 q$, $\iota^4 q$ for q in equation (3), multiplying in order by ι^{-1}, ι^{-2}, ι^{-3}, ι^{-4}, and adding, we have

$$48\sum^{\infty} F(4n+1)\,q^{4n+1} = \sum_{r=1}^{4} \iota^{-r}\left[1+2q^4+2q^{16}+\ldots+\iota^r(2q+2q^9+\ldots)\right]^3,$$

or if $\qquad A = 1+2q^4+2q^{16}+\ldots, \quad B = 2q+2q^9+\ldots$

the right hand is $\sum_{r=1}^{4} \iota^{-r}(A+\iota^r B)^3$, which is easily found to be $12A^2B$. Writing now q for q^4 we have the first result, and similarly for the others.

We also notice that if we write $x = 1/p$, p an integer, in equations (A, B), we can find formulæ for $\sum_r F(m-r^2)$, $\sum_r G(m-r^2)$, where r takes all positive values $\equiv \pm k \bmod p$, where k is given, for which $m-r^2$ is not negative. These formulæ involve finding the coefficient of q^m in expressions such as $\dfrac{\theta_{10}(1/p)}{\theta_{11}(1/p)}\,\theta'_{01}(1/p)\,\theta_{00}$. We may notice that formulæ for expressions of this kind have been found by Hurwitz.†

They involve the coefficients of q^{2n} in the expansion as a power series in q^2 of the integrals of the first kind belonging

* Cf. Hermite, *Collected Works*, vol. ii., pages 109, 240; *Acta Mathematica*, vol. 5, page 297; Kronecker, *Monatsberichte*, 1862, page 309; 1875, page 229.

† See Klein-Fricke, *Modul-functionen*, vol. ii., p. 635.

to the fundamental polygon defined by the linear fractional group of order p. There is no difficulty in identifying products and quotients of theta functions as integrals of the first kind whenever it is possible, but this need not trouble us here.

But we find interesting results of this kind from the expansion of $Q\,\theta_{00}(x, m\omega)$. For instance

$$2\,Q\,\theta_{01}(0, 2\omega) = \sum_{n=0}^{\infty} n\,(-1)^{n+r+1}\,q^{n^2-2r^2}$$

where r takes all integral values from $-\frac{1}{2}(n-1)$ to $\frac{1}{2}n$, both included. From the coefficients of q^m we find

$$2\left[F(m) - 2F(m-2.1^2) + 2F(m-2.2^2) - 2F(m-2.3^2) + \ldots\right]$$

$= \Sigma\, x\,(-1)^{x+y+1}$ where $m = x^2 - 2y^2$, $x > 0$, and y is included in the range $\frac{1}{2}x, -\frac{1}{2}(x-1)$. Similarly

$$2\,Q\,\theta_{00}(0, 2\omega) = -\sum_{m=1}^{\infty} x\,q^m + \tfrac{1}{2}\theta_{00}\,\theta_{00}(0, 2\omega)\,\theta^2_{10}(0, 2\omega).$$

Again

$$F(2m) - 2F(2m-3.1^2) + 2F(2m-3.2^2) - 2F(2m-3.3^2) + \ldots$$

$$= (-1)^{m+1}\,\Sigma x,$$

where now $\qquad x^2 - 3y^2 = m,\;$ with $x > 0$

and $\qquad -\frac{1}{3}(x-1) \leq y \leq \frac{1}{3}x.$

These appear to be results of a new type.

In the formulæ due to Hurwitz, the quadratic form is taken as $ax^2 + hxy + by^2$. A very simple expression for the class number of such forms of given discriminant has been found by Kronecker. I may add that, when the discriminant is negative, I have found Kronecker's expression for the class number in a simple and apparently general way without evaluating expressions such as $\sum_{x,y} (ax^2 + hxy + by^2)^{-\rho}$ wherein $\rho \rightarrow 1$. This investigation is contained in a paper entitled "The class number for definite binary quadratics," which will, I hope, appear in the *Quarterly Journal of Mathematics*.

SOME FORMULÆ IN THE ANALYTIC THEORY OF NUMBERS.

By *S. Ramanujan.*

I HAVE found the following formulæ incidentally in the course of other investigations. None of them seem to be of particular importance, nor does their proof involve the use of any new ideas, but some of them are so curious that they seem to be worth printing. I denote by $d(x)$ the number of divisors of x, if x is an integer, and zero otherwise, and by $\zeta(s)$ the Riemann Zeta-function.

$$(1)\qquad \frac{\zeta^4(s)}{\zeta(2s)} = 1^{-s}d^2(1) + 2^{-s}d^2(2) + 3^{-s}d^2(3) + \ldots,$$

$$(2)\qquad \frac{\eta^4(s)}{(1-2^{-2s})\,\zeta(2s)} = 1^{-s}d^2(1) - 3^{-s}d^2(3) + 5^{-s}d^2(5) - \ldots,$$

where $$\eta(s) = 1^{-s} - 3^{-s} + 5^{-s} - 7^{-s} + \ldots.$$

$$(3)\qquad d^2(1) + d^2(2) + d^2(3) + \ldots + d^2(n)$$

$$= An(\log n)^3 + Bn(\log n)^2 + Cn\log n + Dn + O(n^{\frac{3}{5}+\epsilon}),^*$$

where $$A = \frac{1}{\pi^2},\quad B = \frac{12\gamma - 3}{\pi^2} - \frac{36}{\pi^4}\zeta'(2),$$

γ is Euler's constant, C, D more complicated constants, and ϵ any positive number.

$$(4)\ d^3\left(\frac{n}{1}\right) + d^3\left(\frac{n}{2}\right) + d^3\left(\frac{n}{3}\right) + \ldots = \left\{d\left(\frac{n}{1}\right) + d\left(\frac{n}{2}\right) + d\left(\frac{n}{3}\right) + \ldots\right\}^2 \dagger$$

$$(5)\qquad \sum_1^\infty n^{-s}d^r(n) = \{\zeta(s)\}^{2^r}\phi(s),$$

where $\phi(s)$ is absolutely convergent for $R(s) > \frac{1}{2}$, and in particular

$$(6)\qquad \sum_1^\infty \frac{1}{n^s d(n)} = \prod_p \left\{p^s \log\left(\frac{1}{1-p^{-s}}\right)\right\} = \sqrt{\{\zeta(s)\}}\,\phi(s).$$

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* If we assume the Riemann hypothesis, the error term here is of the form $O(n^{\frac{1}{2}+\epsilon})$.

† Mr. Hardy has pointed out to me that this formula has been given already by Liouville, *Journal de Mathématiques*, ser. 2, vol. 2, 1857, p. 393.

(7) $$\frac{1}{d(1)}+\frac{1}{d(2)}+\frac{1}{d(3)}+\ldots+\frac{1}{d(n)}$$

$$=n\left\{\frac{A_1}{(\log n)^{\frac{1}{2}}}+\frac{A_2}{(\log n)^{\frac{3}{2}}}+\ldots+\frac{A_r}{(\log n)^{r-\frac{1}{2}}}+O\frac{1}{(\log n)^{r+\frac{1}{2}}}\right\},$$

where $$A_1=\frac{1}{\sqrt{\pi}}\prod_p\left\{\sqrt{(p^2-p)}\log\left(\frac{p}{p-1}\right)\right\}$$

and A_2, $A_3 \ldots A_r$ are more complicated constants.

More generally

(8) $$d^s(1)+d^s(2)+d^s(3)+\ldots+d^s(n)$$

$$=n\{A_1(\log n)^{2^s-1}+A_2(\log n)^{2^s-2}+\ldots+A_{2^s}\}+O(n^{\frac{1}{2}+\epsilon}),^{*}$$

if 2^s is an integer, and

(9) $$d^s(1)+d^s(2)+d^s(3)+\ldots+d^s(n)$$

$$=n\left\{A_1(\log n)^{2^s-1}+A_2(\log n)^{2^s-2}+\ldots+\frac{A_{r+2^s}}{(\log n)^r}+O\frac{1}{(\log n)^{r+1}}\right\},$$

if 2^s is not an integer, the A's being constants.

(10) $$d(1)\,d(2)\,d(3)\ldots d(n)=2^{n(\log\log n+C)+\phi(n)},$$

where

$$C=\gamma+\sum_2^\infty\left\{\log_2\left(1+\frac{1}{\nu}\right)-\frac{1}{\nu}\right\}(2^{-\nu}+3^{-\nu}+5^{-\nu}+\ldots).$$

Here 2, 3, 5, ... are the primes and

$$\frac{\phi(n)}{n}=\frac{\gamma-1}{\log n}+\frac{1!}{(\log n)^2}(\gamma+\gamma_1-1)+\frac{2!}{(\log n)^3}(\gamma+\gamma_1+\gamma_2-1)+\ldots$$

$$+\frac{(r-1)!}{(\log n)^r}(\gamma+\gamma_1+\gamma_2+\ldots+\gamma_{r-1}-1)+O\frac{1}{(\log n)^{r+1}},$$

where $$\zeta(1+s)=\frac{1}{s}+\gamma-\gamma_1 s+\gamma_2 s^2-\gamma_3 s^3+\ldots$$

or

$$r!\,\gamma_r=\lim_{\nu\to\infty}\left\{(\log 1)^r+\tfrac{1}{2}(\log 2)^r+\ldots+\frac{1}{\nu}(\log\nu)^r-\frac{1}{r+1}(\log\nu)^{r+1}\right\}.$$

(11) $$d(uv)=\sum_1^\infty\mu(n)\,d\left(\frac{u}{n}\right)d\left(\frac{v}{n}\right)=\Sigma\,\mu(\delta)\,d\left(\frac{u}{\delta}\right)d\left(\frac{v}{\delta}\right),$$

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* Assuming the Riemann hypothesis.

where δ is a common factor of u and v, and

$$\frac{1}{\zeta(s)} = \sum_1^\infty \frac{\mu(n)}{n^s}.$$

If $$D_v(n) = d(v) + d(2v) + \ldots + d(nv),$$

we have

(12) $$D_v(n) = \Sigma\, \mu(\delta)\, d\left(\frac{v}{\delta}\right) D_1\left(\frac{n}{\delta}\right),$$

where δ is a divisor of v, and

(13) $$D_v(n) = \alpha(v)\, n(\log n + 2\gamma - 1) + \beta(v)\, n + \Delta_v(n),$$

where $$\sum_1^\infty \frac{\alpha(\nu)}{\nu^s} = \frac{\zeta^2(s)}{\zeta(1+s)}, \quad \sum_1^\infty \frac{\beta(\nu)}{\nu^s} = -\frac{\zeta^2(s)\,\zeta'(1+s)}{\zeta^2(1+s)},$$

and $$\Delta_v(n) = O(n^{\frac{1}{3}} \log n).*$$

(14) $$d(v+c) + d(2v+c) + d(3v+c) + \ldots + d(nv+c)$$
$$= \alpha_c(v)\, n(\log n + 2\gamma - 1) + \beta_c(v)\, n\, \Delta_{v,c}(n),$$

where $$\sum_1^\infty \frac{\alpha_c(\nu)}{\nu^s} = \frac{\zeta(s)\,\sigma_{-s}(|c|)}{\zeta(1+s)},$$

$$\sum_1^\infty \frac{\beta_c(\nu)}{\nu^s} = -\frac{\zeta(s)\,\sigma_{-s}(|c|)}{\zeta(1+s)} \left\{\frac{\zeta'(s)}{\zeta(s)} + \frac{\zeta'(1+s)}{\zeta(1+s)} + \frac{\sigma_{-s}'(|c|)}{\sigma_{-s}(|c|)}\right\},$$

$\sigma_s(n)$ being the sum of the sth powers of the divisors of n and $\sigma'_s(n)$ the derivative of $\sigma_s(n)$ with respect to s, and

$$\Delta_{v,c}(n) = O(n^{\frac{1}{3}} \log n)\dagger.$$

The formulæ (1) and (2) are special cases of

(15) $$\frac{\zeta(s)\,\zeta(s-a)\,\zeta(s-b)\,\zeta(s-a-b)}{\zeta(2s-a-b)}$$
$$= 1^{-s}\sigma_a(1)\,\sigma_b(1) + 2^{-s}\sigma_a(2)\,\sigma_b(2) + 3^{-s}\sigma_a(3)\,\sigma_b(3) + \ldots;$$

(16) $$\frac{\eta(s)\,\eta(s-a)\,\eta(s-b)\,\eta(s-a-b)}{(1-2^{-2s+a+b})\,\zeta(2s-a-b)}$$
$$= 1^{-s}\sigma_a(1)\,\sigma_b(1) - 3^{-s}\sigma_a(3)\,\sigma_b(3) + 5^{-s}\sigma_a(5)\,\sigma_b(5) - \ldots.$$

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* It seems not unlikely that $\Delta_v(n)$ is of the form $O(n^{\frac{1}{4}+\epsilon})$. Mr. Hardy has recently shown that $\Delta_1(n)$ is not of the form $o\{(n \log n)^{\frac{1}{4}} \log\log n\}$. The same is true in this case also.

† It is very likely that the order of $\Delta_{v,c}(n)$ is the same as that of $\Delta_1(n)$.

It is possible to find an approximate formula for the general sum

(17) $\sigma_a(1)\sigma_b(1)+\sigma_a(2)\sigma_b(2)+\ldots+\sigma_a(n)\sigma_b(n)$.

The general formula is complicated. The most interesting cases are $a=0$, $b=0$, when the formula is (3); $a=0$, $b=1$, when it is

$$\frac{\pi^2 n^2}{72\,\zeta(3)}(\log n+2c)+nE(n), \tag{18}$$

where $$c=\gamma-\tfrac{1}{4}+\frac{\zeta'(2)}{\zeta(2)}-\frac{\zeta'(3)}{\zeta(3)},$$

and the order of $E(n)$ is the same as that of $\Delta_1(n)$; and $a=1$, $b=1$, when it is

$$\tfrac{5}{6}n^3\zeta(3)+E(n), \tag{19}$$

where $E(n)=O\{n^2(\log n)^2\}$, $E(n)\neq o(n^2\log n)$.

If $s>0$, then

$$\sigma_s(1)\,\sigma_s(2)\,\sigma_s(3)\,\sigma_s(4)\ldots\sigma_s(n)=\theta c^n(n!)^s, \tag{20}$$

where $1>\theta>(1-2^{-s})(1-3^{-s})(1-5^{-s})\ldots(1-\varpi^{-s})$,

ϖ is the greatest prime not exceeeding n, and

$$c=\prod_p\left\{\left(\frac{p^{2s}-1}{p^{2s}-p^s}\right)^{1/p}\left(\frac{p^{3s}-1}{p^{3s}-p^s}\right)^{1/p^2}\left(\frac{p^{4s}-1}{p^{4s}-p^s}\right)^{1/p^3}\cdots\right\}.$$

If

$$(\tfrac{1}{2}+q+q^4+q^9+q^{16}+\ldots)^2=\tfrac{1}{4}+\sum_1^\infty r(n)\,q^n,$$

so that $$\zeta(s)\,\eta(s)=\sum_1^\infty r(n)\,n^{-s},$$

then

$$\frac{\zeta^2(s)\,\eta^2(s)}{(1+2^{-s})\,\zeta(2s)}=1^{-s}r^2(1)+2^{-s}r^2(2)+3^{-s}r^2(3)+\ldots. \tag{21}$$

$$r^2(1)+r^2(2)+r^2(3)+\ldots+r^2(n)=\frac{n}{4}(\log n+C)+O(n^{\frac{3}{5}+\epsilon}), \tag{22}$$

where

$$C=4\gamma-1+\tfrac{1}{3}\log 2-\log\pi+4\log\Gamma(\tfrac{3}{4})-\frac{12}{\pi^2}\zeta'(2).$$

These formulæ are analogous to (1) and (3).

AN INTERPRETATION OF PENTASPHERICAL COORDINATES.

By *T. C. Lewis, M.A.*

1. In Cartesian coordinates the position of a point is at once determined by its coordinates; but if the system of coordinates due to M. Gaston Darboux is employed (called pentaspherical in 3-space, and by extension available for n-space in general) the position of a point when its coordinates are given has not, so far as I know, been investigated so as to lead to a similarly easy geometrical determination. This can however be done.

2. It is known (*Messenger of Mathematics*, vol. xliv., 1915, p. 161) that if $x, y, z, \ldots,$ be the rectangular Cartesian coordinates of a point, and $x_1, x_2, x_3, \ldots, x_{n+2}$ its Darboux coordinates; and if $a_k, b_k, \ldots$ be the Cartesian coordinates of the k^{th} vertex of reference of the Darboux system, then

$$2x + \Sigma \frac{a_k x_k}{\rho_k} = 0,$$

$$2y + \Sigma \frac{b_k x_k}{\rho_k} = 0,$$

$$\&\text{c.}, \qquad \&\text{c.}$$

Therefore $$x \Sigma \frac{x_k}{\rho_k} = \Sigma \frac{a_k x_k}{\rho_k},$$

$$\&\text{c.}, \qquad \&\text{c.}$$

Therefore the point is the centre of gravity of masses proportional to x_k/ρ_k at the k^{th} vertex, k having all values from 1 to $n+2$. In other words, the point is the mean centre of the vertices for a system of multiples

$$x_1/\rho_1, \; x_2/\rho_2, \; \ldots, \; x_{n+2}/\rho_{n+2}.$$

The position of the point is therefore determined by a simple geometrical construction.

3. If it is desired to find the system of multiples corresponding to only $n+1$ of the vertices, this may be done; for

any point whatever may be regarded as the mean centre of the $n+2$ vertices for the system of multiples

$$x_{n+2}\rho_{n+2}/\rho_1^2,\quad x_{n+2}\rho_{n+2}/\rho_2^2,\ \&c., \&c.,$$

since the sum of these multiples is zero.

Therefore the system of multiples for the $n+1$ vertices from the first to the $(n+1)^{\text{th}}$ is

$$(\rho_1 x_1 - \rho_{n+2}x_{n+2})\,\rho_1^2,\quad (\rho_2 x_2 - \rho_{n+2}x_{n+2})\,\rho_2^2,\ \&c., \&c.$$

4. This leads to an equally simple method for determining geometrically the centre of a circle or n-sphere.

If the equation of the circle, &c., be given in its general form

$$\Sigma\, \alpha_k x_k = 0,$$

the coordinates of the centre are given by

$$x'_k = \rho\,(\rho - 2\alpha_k\rho_k)/\rho_k,$$

where ρ is the radius of the circle or n-sphere.

The system of multiples at the vertices for which the centre of the circle, &c., is the mean centre is therefore

$$\rho^2/\rho_1^2 - 2\alpha_1\rho/\rho_1,\quad \rho^2/\rho_2^2 - 2\alpha_2\rho/\rho_2,\ \&c.$$

Since $$\frac{1}{\rho_1^2} + \frac{1}{\rho_2^2} + \dots + \frac{1}{\rho^2_{n+2}} = 0,$$

this is equivalent to the system

$$\alpha_1/\rho_1,\quad \alpha_2/\rho_2,\ \dots,\ \alpha_{n+2}/\rho_{n+2}.$$

Thus the centre is at once determined.

For instance, take the equation to the circle

$$\rho_1 x_1 + \rho_2 x_2 + \rho_3 x_3 = 0.$$

The centre is seen to be at the centre of gravity of the triangle of reference.

A CIRCLE SIMPLY RELATED TO A TRIANGLE.

By *T. C. Lewis, M.A.*

1. LET ABC be a triangle, P the orthocentre.

On AD, BE, CF, the perpendiculars from the angular points on the opposite sides, measure distances AH, BK, CL equal to ρ_1, ρ_2, ρ_3 respectively, *i.e.*

$$AH^2 = \rho_1^2 = \tfrac{1}{2}(b^2 + c^2 - a^2) = bc \cos A\,; \ \&c.$$

Take points at one third of the distance from A to K, and from A to L, from B to H and from B to L, from C to H and C to K.

A circle passes through these six points, and its centre is at G, the centre of gravity at the triangle. It has interesting geometrical properties.

Through B and C draw lines parallel to AC, AB, meeting at A'.

Then
$$\begin{aligned} A'K^2 &= \tfrac{1}{2}(a^2 + b^2 + c^2) \\ &= \rho_1^2 + \rho_2^2 + \rho_3^2 \\ &= A'L^2. \end{aligned}$$

Therefore a circle with centre at A', the square of whose radius is $\rho_1^2 + \rho_2^2 + \rho_3^2$, passes through the points K, L.

Now $AA' = 3AG$.

Therefore A is a centre of similitude of this circle, with centre at A', and of a circle with centre at G the square of whose radius, ρ, is $\tfrac{1}{9}(\rho_1^2 + \rho_2^2 + \rho_3^2)$. And the latter circle will pass through the points at one third of the distance from A to K and to L.

Similarly it passes through the remaining four of the six points mentioned. Therefore the proposition is established.

The six points determine a hexagon, three alternate sides of which are parallel to, and two thirds of the length of the sides of the triangle ABC; while the remaining sides are parallel to, and one third of the length of the sides of the triangle HKL.

Six other points on the circle are similarly determined if AH, BK, CL be taken in the opposite direction.

The remaining centre of similitude of the two circles is at D', the middle point of BC. Therefore if any one of the four points K, L be joined to D', and produced to a point whose distance from D' is one third of the joining line, such point

also lies on the circle with the centre at G. Thus altogether twenty-four points on the circle are determined, and the circle might be called the 24-point circle.

2. The square of the tangent from any angular point A is

$$AG^2 - \rho^2 = \tfrac{1}{3}\rho_1^2$$

whence the intersections with the sides can be determined.

So the product of the segments of a chord through P is $AP.PD$, or $-\rho_4^2$.

3. Using Darboux coordinates the equation to the circle is

$$\rho_1 x_1 + \rho_2 x_2 + \rho_3 x_3 = 0,$$

for this circle has its centre at G and its radius equal to ρ.

So also the circle with centre A' passing through K and L is

$$-\rho_1 x_1 + \rho_2 x_2 + \rho_3 x_3 = 0.$$

The interpretation of the former equation is that the circle is the locus of a point Q such that

$$\begin{aligned} QA^2 + QB^2 + QC^2 &= \rho_1^2 + \rho_2^2 + \rho_3^2 \\ &= \tfrac{1}{2}(a^2 + b^2 + c^2) \\ &= 9\rho^2. \end{aligned}$$

It may be noted that the circumscribing and nine-point circles have equations and corresponding geometrical relations which are not so simple. Their equations respectively are

$$\rho_1 x_1 + \rho_2 x_2 + \rho_3 x_3 - \rho_4 x_4 = 0$$

and

$$\rho_1 x_1 + \rho_2 x_2 + \rho_3 x_3 + \rho_4 x_4 = 0,$$

and they are the loci of a point Q such that for the circumscribing circle

$$QA^2 + QB^2 + QC^2 - QP^2 = \rho_1^2 + \rho_2^2 + \rho_3^2 - \rho_4^2,$$

and for the nine-point circle

$$\begin{aligned} QA^2 + QB^2 + QC^2 + QP^2 &= \rho_1^2 + \rho_2^2 + \rho_3^2 + \rho_4^2 \\ &= 4R^2 \\ &= 16R'^2 \end{aligned}$$

when R, R' are the radii of the circumscribing and nine-point circles.

THE BROCARD AND LEMOINE CIRCLES.

By *T. C. Lewis, M.A.*

1. If K is the isogonal conjugate of the centroid of a triangle ABC, and O the centre of the circumscribed circle, the Brocard circle is the circle on OK as diameter.

The equation to the circumcircle in Darboux coordinates is

$$\rho_1 x_1 + \rho_2 x_2 + \rho_3 x_3 - \rho_4 x_4 = 0,$$

and its centre (x_1', x_2', x_3', x_4') is determined by

$$R^2 = \rho_1 x_1' + \rho_1^2 = \rho_2 x_2' + \rho_2^2 = \rho_3 x_3' + \rho_3^2 = \rho_4 x_4' - \rho_4^2.$$

K is the mean centre of A, B, C for the system of multiples a^2, b^2, c^2. It is therefore the centre of the circle

$$a^2\rho_1 x_1 + b^2\rho_2 x_2 + c^2\rho_3 x_3 = 0.$$

Let R' be the radius of this circle, then

$$R'^2 = \frac{a^4\rho_1^2 + b^4\rho_2^2 + c^4\rho_3^2}{(a^2 + b^2 + c^2)^2}$$

$$4R'^2 = \frac{\rho_1^2(\rho_2^2 + \rho_3^2)^2 + \rho_2^2(\rho_3^2 + \rho_1^2)^2 + \rho_3^2(\rho_1^2 + \rho_2^2)^2}{(\rho_1^2 + \rho_2^2 + \rho_3^2)^2}$$

$$= \frac{(\rho_1^2 + \rho_2^2 + \rho_3^2)(\rho_2^2\rho_3^2 + \rho_3^2\rho_1^2 + \rho_1^2\rho_2^2) + 3\rho_1^2\rho_2^2\rho_3^2}{(\rho_1^2 + \rho_2^2 + \rho_3^2)^2}$$

$$= \frac{-\rho_1^2\rho_2^2\rho_3^2}{(\rho_1^2 + \rho_2^2 + \rho_3^2)^2\rho_4^2}(\rho_1^2 + \rho_2^2 + \rho_3^2 - 3\rho_4^2),$$

$$R'^2 = \frac{-\rho_1^2\rho_2^2\rho_3^2}{(\rho_1^2 + \rho_2^2 + \rho_3^2)^2\rho_4^2}(R^2 - \rho_4^2).$$

But the perpendiculars from K on the sides of the triangle are proportional to those sides. Let the perpendicular on BC be ka. Then

$$k = \frac{2\Delta}{(a^2 + b^2 + c^2)},$$

therefore
$$k^2 = \frac{\Delta^2}{(\rho_1^2+\rho_2^2+\rho_3^2)^2}$$
$$= \frac{-\rho_1^2\rho_2^2\rho_3^2}{4(\rho_1^2+\rho_2^2+\rho_3^2)^2\rho_4^2}.$$

Therefore
$$R'^2 = 4k^2(R^2-\rho_4^2).$$

Also
$$OK^2 - R'^2 = \frac{a^2\rho_1 x_1' + b^2\rho_2 x_2' + c^2\rho_3 x_3'}{a^2+b^2+c^2}$$
$$= R^2 - \frac{a^2\rho_1^2 + b^2\rho_2^2 + c^2\rho_3^2}{a^2+b^2+c^2},$$

therefore
$$OK^2 = R^2 + R'^2 - \frac{\rho_1^2(\rho_2^2+\rho_3^2)+\rho_2^2(\rho_3^2+\rho_1^2)+\rho_3^2(\rho_1^2+\rho_2^2)}{2(\rho_1^2+\rho_2^2+\rho_3^2)}$$
$$= R^2 + 4k^2(R^2-\rho_4^2) + \frac{\rho_1^2\rho_2^2\rho_3^2}{(\rho_1^2+\rho_2^2+\rho_3^2)^2\rho_4^2}(4R^2-\rho_4^2)$$
$$= (1-12k^2)R^2,$$

The equation to the circle on OK as diameter therefore is
$$\tfrac{1}{2}(\rho_1 x_1+\rho_2 x_2+\rho_3 x_3-\rho_4 x_4) + R^2 + \frac{a^2\rho_1 x_1 + b^2\rho_2 x_2 + c^2\rho_3 x_3}{a^2+b^2+c^2} + R'^2$$
$$= R^2 + R'^2 - 4k^2(4R^2-\rho_4^2),$$

or $\rho_1 x_1 + \rho_2 x_2 + \rho_3 x_3 - \rho_4 x_4$
$$+ \frac{a^2\rho_1 x_1 + b^2\rho_2 x_2 + c^2\rho_3 x_3}{\rho_1^2+\rho_2^2+\rho_3^2} - \frac{2\rho_1^2\rho_2^2\rho_3^2}{(\rho_1^2+\rho_2^2+\rho_3^2)\rho_4^2} = 0.$$

Making this homogeneous it becomes
$$\rho_1 x_1 + \rho_2 x_2 + \rho_3 x_3 - \rho_4 x_4 + \frac{a^2\rho_1 x_1 + b^2\rho_2 x_2 + c^2\rho_3 x_3}{\rho_1^2+\rho_2^2+\rho_3^2}$$
$$+ \frac{\rho_1^2\rho_2^2\rho_3^2}{(\rho_1^2+\rho_2^2+\rho_3^2)\rho_4^2}\left(\frac{x_1}{\rho_1}+\frac{x_2}{\rho_2}+\frac{x_3}{\rho_3}+\frac{x_4}{\rho_4}\right) = 0,$$

which reduces to
$$\rho_1 x_1 + \rho_2 x_2 + \rho_3 x_3 - \rho_4 x_4 = \frac{\rho_1^2\rho_2^2\rho_3^2}{\rho_1^2+\rho_2^2+\rho_3^2}\left(\frac{x_1}{\rho_1^3}+\frac{x_2}{\rho_2^3}+\frac{x_3}{\rho_3^3}-\frac{x_4}{\rho_4^3}\right).$$

From this it appears that the Brocard circle passes through the intersection of the circumcircle and a circle whose centre is K which cuts the circumcircle orthogonally, this last circle being, however, unreal.

2. If LK be drawn through K parallel to BC, and meet AB in L, the Lemoine circle is the locus of a point such that the sum of the squares of its distances from O and K is constant, viz. the same as for the point L.

Now
$$OL^2 = R^2 - AL.LB$$
$$= R^2 - \frac{a^2(b^2+c^2)c^2}{(a^2+b^2+c^2)^2},$$

and
$$LK^2 = \frac{a^2c^4}{(a^2+b^2+c^2)^2},$$

therefore
$$OL^2 + LK^2 = R^2 - \frac{a^2b^2c^2}{(a^2+b^2+c^2)^2}$$
$$= (1-4k^2)R^2.$$

Therefore the equation to the Lemoine circle is

$$\tfrac{1}{2}(\rho_1x_1+\rho_2x_2+\rho_3x_3-\rho_4x_4) + \frac{a^2\rho_1x_1+b^2\rho_2x_2+c^2\rho_3x_3}{2(\rho_1^2+\rho_2^2+\rho_3^2)} + R^2 + R'^2$$
$$= (1-4k^2)R^2,$$

that is, when rendered homogeneous,

$$\rho_1x_1+\rho_2x_2+\rho_3x_3-\rho_4x_4+\frac{a^2\rho_1x_1+b^2\rho_2x_2+c^2\rho_3x_3}{\rho_1^2+\rho_2^2+\rho_3^2}$$
$$-2k^2(\rho_1^2+\rho_2^2+\rho_3^2-\rho_4^2)\left(\frac{x_1}{\rho_1}+\frac{x_2}{\rho_2}+\frac{x_3}{\rho_3}+\frac{x_4}{\rho_4}\right)=0,$$

where the distance of the symmedian point from any side is k times the length of that side, this value having been already determined.

SUCCESSIVE TRANSFORMS OF AN OPERATOR WITH RESPECT TO A GIVEN OPERATOR.

By *G. A. Miller.*

LET s_0 and t be any two non-commutative operators, and let $s_1, s_2, \ldots, s_n$ represent the n commutators obtained as follows:

$$t^{-1}s_0t=s_1s_0,\quad t^{-1}s_1t=s_2s_1,\quad \ldots,\quad t^{-1}s_{n-1}t=s_ns_{n-1}.$$

If these n commutators are all commutative with each other then it results directly that

$$t^{-n}s_0t^n=s_n\, s_{n-1}^n\, s_{n-2}^{\frac{1}{2}\{n(n-1)\}}\ldots s_1^n s_0,$$

where the exponents of the symbols in the second member are the binominal coefficients in order.

If t^α is any power of t which is commutative with s_0 it results that

$$s_\alpha\, s_{\alpha-1}^\alpha\, s_{\alpha-2}^{\frac{1}{2}\{\alpha(\alpha-1)\}}\ldots s_1^\alpha=1.$$

Hence it follows that

$$t^{-(\alpha+\beta)}s_0\,t^{\alpha+\beta}=s_\beta\, s_{\beta-1}^\beta\, s_{\beta-2}^{\frac{1}{2}\{\beta(\beta-1)\}}\ldots s_1^\beta s_0,$$

and that
$$s_{\alpha+\beta}\, s_{\alpha+\beta-1}^\alpha\, s_{\alpha+\beta-2}^{\frac{1}{2}\{\alpha(\alpha-1)\}}\ldots s_{\beta+1}^\alpha=1.$$

When the commutators $s_1, s_2, \ldots, s_n$ are not assumed to be commutative the formulæ which correspond to those just found become much more complex. To find such a formula by induction we may proceed as follows:

$$t^{-1}s_0t=s_1s_0,$$
$$t^{-2}s_0t^2=s_2s_1{}^2s_0,$$
$$t^{-3}s_0t^3=s_3s_2{}^2s_1\,.\,s_2s_1{}^2s_0,$$
$$t^{-4}s_0t^4=s_4s_3{}^2s_2(s_3s_2{}^2s_1)^2s_2s_1{}^2s_0,$$
$$t^{-4}s_0t^5=s_5s_4{}^2s_3(s_4s_3{}^2s_2)^2s_3s_2{}^2s_1\,.\,s_4s_3{}^2s_2(s_3s_2{}^2s_1)^2s_2s_1{}^2s_0.$$

It is evident that each of these five transforms is the product of two expressions which differ from each other only as regards the subscripts. In the former of these two factors each subscript is equal to the corresponding subscript of the latter increased by unity. The number of the linear factors in the α^{th} transform is evidently 2^α. As each of the two factors of the α^{th} transform is similar to the $(\alpha-1)^{\text{th}}$ transform,

the $(\alpha+1)^{\text{th}}$ transform can be deduced directly from the α^{th} transform according to the law involved in deriving the α^{th} transform from the $(\alpha-1)^{\text{th}}$. Hence the following rule:—*To obtain the n^{th} transform multiply the $(n-1)^{\text{th}}$ transform on the left by the expression obtained by increasing each subscript of this transform by unity.* When n is even the first half of the former of these two expressions and the last half of the latter are identical, in order, and hence this part may be written in the form of a square.

From the given rule to write down the n^{th} transform if the $(n-1)^{\text{th}}$ transform is given, it is easy to derive the following rule to find the n^{th} transform directly: when $n>5$ write the expression $s_n s^2_{n-1} s_{n-2}$, then multiply it on the right by the square of the expression obtained by diminishing each of the subscripts in $s_n s^2_{n-1} s_{n-2}$ by unity, then multiply this product on the right by the result obtained by diminishing each of the subscripts in $s_n s^2_{n-1} s_{n-2}$ by 2. The product thus obtained is again multiplied on the right by the square of the result obtained by diminishing each of its subscripts by unity, and this latter result is multiplied on the right by the expression obtained by diminishing each of these subscripts by 2. If s_0 has not been reached by these operations, the last product is to be treated in exactly the same manner as the preceding product was treated, and the operations are repeated until s_0 is reached. The expression in which s_0 occurs is never squared even if the above rule would require that this factor be squared.

This rule clearly gives rise to an expression consisting of two factors which are such that the second can be obtained by merely diminishing each of the subscripts of the first by unity, and if this expression is the n^{th} transform the second of these factors is the $(n-1)^{\text{th}}$ transform. Hence this rule is equivalent to the one given above. It should be observed that even the first factor $s_n s^2_{n-1} s_{n-2}$ is formed from s_n according to this rule, and that when $n=1$ this rule gives $s_1 s_0$, when $n=2$ it gives $s_2 s_1^2 s_0$, when $n=3$ it gives $s_3 s_2^2 s_1 . s_2 s_1^2 s_0$, when $n=4$ it gives $s_4 s_3^2 s_2 (s_3 s_2^2 s_1)^2 s_2 s_1 s_0^2$, etc.

The n commutators $s_1,\ s_2,\ \ldots,\ s_n$ cannot be independent unless the index m of the lowest power of t which is commutative with s_0 exceeds n, since

$$s_m s_m^2 s_{m-2} (s_{m-1} s^2_{m-2} s_{m-3})^2 s_{m-2} s^2_{m-3} s_{m-4} \ldots s_1^2 = 1.$$

In particular, when $m=2$ it results that

$$s_2 = s_1^{-2},\ s_3 = s_2^{-2},\ \ldots,\ s_n = s_{n-1}^{-2}.$$

Hence these commutators generate a cyclic group whenever t^2 is commutative with s_0. If s_0^2 is also commutative with t then s_0 must also transform this cyclic group into itself, and hence we have the known results that if the square of each of two operators is commutative with the other operator these operators generate a group whose commutator is cyclic.

The successive transforms of an operator have been employed frequently, especially in regard to prime power groups where each of the n commutators $s_1, s_2, \ldots, s_n$ may be assumed to be contained in a smaller group than the preceding, with the exception of the first of these commutators which is contained in an invariant subgroup of the original group. The special formula when each of these n commutators is commutative with all of the others is also known, but the general rule of finding the n^{th} transform and the method of proof here outlined are supposed to be new.

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SYSTEMS OF PARTICLES EQUIMOMENTAL WITH A UNIFORM TETRAHEDRON.

By *Eric H. Neville.*

Simple systems of particles equimomental with a uniform tetrahedron have long been known, but the methods given in the standard text-books for demonstrating their property leave much to be desired. Though as far as I am concerned original, the following method may well have been known to the teachers of the last generation, but there is evidence to the contrary in its absence from the pages of Routh.

Let PQ, RS be opposite edges of a uniform tetrahedron of mass M, and let their mid-points be U, V and their lengths $2a$, $2b$; let the length of UV be $2c$, and let G be the mid-point of UV. A plane parallel to PQ and RS, cutting UV in a point whose distance from G towards U is ct, cuts the surface of the tetrahedron in a parallelogram of sides $a(1+t)$, $b(1-t)$, with angles independent of t. Since a parallelogram of mass m is equimomental with particles of masses $m/12$ at the vertices and a particle of mass $2m/3$ at the centre, the

tetrahedron is equimomental with a distribution of varying line-density along the five lines PR, PS, QR, QS, UV, the density in each line being proportional to $1-t^2$, and the total mass of each of the first four lines being $M/12$ and of the fifth line being $2M/3$. Since

$$\int_{-1}^{1} k(1-t^2)\,dt = 4k/3, \quad \int_{-1}^{1} k(1-t^2)\,t^2\,dt = 4k/15,$$

the part of a line of density $k(1-t^2)$ which corresponds to values of t between -1 and 1 has mass n if k is equal to $3n/4$, and the line is equimomental with three particles, one of mass $n/10$ at each end and one of mass $4n/5$ at the mid-point. It follows at once that

A uniform tetrahedron of mass M is equimomental with a system of eleven particles, one of mass $M/60$ at each vertex, one of mass $M/15$ at the mid-point of each edge, and one of mass $8M/15$ at the centroid.

The deduction of the familiar systems with five particles, of which four are at the vertices, and with seven particles, of which six are at the mid-points of the edges, requires only applications of the theorem that a system of three equal particles of mass $m/3$ at the mid-points of the sides of a triangle is equimomental with a system of four particles, one of mass $m/12$ at each vertex and one of mass $3m/4$ at the centroid.

It is evident that the method used here is applicable to many other problems, and it is interesting to use it in the case of a triangle. A uniform line of mass m is equimomental with particles of mass $m/6$ at its end-points and a particle of mass $2m/3$ at its mid-point, and the integrations of t, t^2, and t^3 from 0 to 1 are sufficient to show that a line PQ of mass n whose density is proportional to distance from P has its centroid at the point of trisection nearer to Q and is equimomental with three particles, one of mass $n/12$ at P, one of mass $n/6$ at Q, and one of mass $3n/4$ at the centroid. It follows that a triangle ABC of mass M is equimomental with a system of seven particles, one of mass $M/12$ at A, two of mass $M/36$ at B and C, one of mass $M/9$ at the mid-point of BC, two of mass $M/8$ at the points of trisection of AB, AC which are the further from A, and one of mass $M/2$ at the centroid of the triangle. Superposing three distributions of this form each with total mass $M/3$, we find a symmetrical system composed of thirteen particles, one of mass $5M/108$ at each vertex, one of mass $M/27$ at the mid-

point of each side, one of mass $M/24$ at each point of trisection of each side, and one of mass $M/2$ at the centroid, and this system can be replaced immediately by a system of seven particles, one of mass $M/18$ at each vertex, one of mass $M/9$ at the mid-point of each side, and one of mass $M/2$ at the centroid.

NOTE ON AN ELIMINATION.

By *Prof. E. J. Nanson.*

PROFESSOR Steggall having recently, *Messenger*, vol. xiiv., p. 111, recalled attention to a verification by Cayley, that if $a+b+c=0$ and $x+y+z=0$, then

$$4(\Sigma ax)^3 - 3\Sigma ax \,.\, \Sigma a^2 \,.\, \Sigma x^2 - 2\Pi(b-c)(y-z) - 54\,abcxyz = 0,$$

reference may be made to a proof by Leudesdorf, *Messenger*, vol. xii., p. 176.

The following verification, although not so elegant as those of Leudesdorf and Steggall, may also be put on record.

Since $a+b+c=0$ and $x+y+z=0$ we may take a, b, c to be the roots of $X^3+qX+r=0$ and put $x=\lambda a+\mu(b-c)$, &c., so that $y-z=\lambda(b-c)-3\mu a$, &c. Then, since $\Sigma a(a-b)(a-c)=-9r$ and $\Sigma bc(b-c)=-\delta$, where $\delta=\Pi(b-c)$, so that $-\delta^2=4q^3+27r^2$, we have

$$xyz = -r\lambda^3 - \delta\lambda^2\mu + 9r\lambda\mu^2 + \delta\mu^3,$$

$$\Pi(y-z) = \quad \delta\lambda^3 - 27r\lambda^2\mu - 9\delta\lambda\mu^2 + 27r\mu^3,$$

so that

$$\Pi(b-c)(y-z) + 27abcxyz = \lambda(\lambda^2-9\mu^2)(\delta^2+27r^2).$$

Also $$\Sigma ax = \lambda\,\Sigma a^2 = -2q\lambda,$$

and $$\Sigma x^2 = \lambda^2\,\Sigma a^2 + \mu^2\,\Sigma(b-c)^2 = -2(\lambda^2+\mu^2)\,q,$$

so that $$4(\Sigma ax)^3 - 3\Sigma ax \,.\, \Sigma a^2 \,.\, \Sigma x^2 = -8\lambda(\lambda^2-9\mu^2)\,q^3.$$

Thus the relation to be proved is seen to be true because

$$\delta^2 + 27r^2 + 4q^3 = 0.$$

Melbourne,
August 9th, 1915.

TIME AND ELECTROMAGNETISM.

By *Prof. H. Bateman.*

The interval between two moving points.

§ 1. For descriptive purposes a system of rectangular coordinates (x, y, z) and a time variable t will be used to express the ideas of motion and the propagation of light in mathematical language, but the observers whose experiences we are about to discuss are supposed to have no direct knowledge of this system of coordinates. To fix ideas we shall also assume that with the above system of coordinates the velocity of light is the constant quantity c and is independent of the motion of the source and the state of the observer.* It will be convenient to regard this system of coordinates as the standard system, and to define motion in the usual way as motion relative to the axes of coordinates.

Now consider two observers, A and B, each of whom is provided with an ideal clock which can be regulated so as to indicate at time t any arbitrarily chosen continuous monotonic function of t. The two observers are supposed to have no direct means of ascertaining whether they are moving relatively to one another or not. They are supposed to set their clocks so that they are 'together' according to Einstein's criterion†, and the problem is to find what function of t each clock must indicate in order that the criterion may be satisfied when the two observers are moving relatively to our standard set of axes in an arbitrary manner, which may be specified as follows:

$$\left.\begin{array}{lll} (A) \qquad x=x(t), & y=y(t), & z=z(t) \\ (B) \qquad x=\xi(t), & y=\eta(t), & z=\zeta(t) \end{array}\right\} \quad \ldots\ldots\ldots\ldots(1).$$

We shall suppose, however, that each observer is always moving with a velocity which is less than that of light so that at any instant τ, B sees only one position‡ of A by means of light sent out from A at time t_1, and the light sent out from B

* The foundations of an optical geometry of space and time, in which the above condition is satisfied, have been laid by A. A. Robb, *A Theory of Space and Time*, Camb. Univ. Press (1914).

† *Ann. of Phys.* Bd. 17 (1905), pp. 891–921.

‡ This follows from a theorem due to Liénard, *L'éclairage électrique*, t. 16 (1898), p. 5. See also H. Bateman, *The physical aspect of Time*, Manchester Memoirs (1910); *Electrical and Optical Wave Motion*, Camb. Univ. Press (1915), ch. 8, p. 116; A. W. Conway, *Proc. Lond. Math. Soc.* ser. 2, vol. 1. (1903).

at the instant τ reaches A at only one instant t_2. The two instants t_1, t_2, which satisfy the inequality $t_1 < \tau < t_2$, are the two real roots of an equation $F(t, \tau) = 0$, which, according to the usual theory of light, is

$$[x(t) - \xi(\tau)]^2 + [y(t) - \eta(\tau)]^2 + [z(t) - \zeta(\tau)]^2 = c^2(t - \tau)^2 \quad \ldots (2).$$

Let A's clock indicate at time t the number $f(t)$ and B's clock the number $\phi(t)$, then A's clock will be said to be *running uniformly* with reference to* B if

$$f(t_2) - f(t_1) = 2T_{ab} \quad \ldots\ldots (3),$$

where T_{ab} is a constant. This means that a signal always takes the same clock-time to go from A to B and back again. If now $\phi(\tau)$ be defined by the equations

$$f(t_2) - T_{ab} = \phi(\tau) = f(t_1) + T_{ab} \quad \ldots\ldots (4),$$

the two clocks will be '*synchronous*' or '*together*.'

It is easy to see that the function $\phi(\tau)$ satisfies an equation analogous to (3), for if τ and τ' are the two real roots of the equation $F(t_2, \tau) = 0$, we have

$$\phi(\tau') = f(t_2) + T_{ab},$$

consequently
$$\phi(\tau') - \phi(\tau) = 2T_{ab} \quad \ldots\ldots (5).$$

Hence whenever a function of type $f(t)$ exists there is a quantity T_{ab} symmetrically related to the two moving points A and B, which remains constant during the motion. This quantity will be called the *interval* between the two moving points.

Determination of the interval in a particular case.

§ 2. Let us consider the case when the movements of the two observers A and B are specified by the equations

$$\left.\begin{array}{ll} (A) & x = l(a + ut), \quad y = m(a + ut), \quad z = n(a + ut) \\ (B) & x = \lambda(a + ut), \quad y = \mu(a + ut), \quad z = \nu(a + ut) \end{array}\right\} \ldots (6),$$

where $l, m, n, \lambda, \mu, \nu, a, u$ are constants. Equation (2) then gives

$$\left.\begin{array}{l} a + ut_1 = \dfrac{a + u\tau}{P}\left[R - (R^2 - PQ)^{\frac{1}{2}}\right] \\ a + ut_2 = \dfrac{a + u\tau}{P}\left[R + (R^2 - PQ)^{\frac{1}{2}}\right] \end{array}\right\} \ldots\ldots (7),$$

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* Cf. E. V. Huntingdon, *Phil. Mag.* April (1912), for the case in which the two observers are not moving relatively to one another.

where

$$P = \frac{c^2}{u^2} - l^2 - m^2 - n^2, \quad Q = \frac{c^2}{u^2} - \lambda^2 - \mu^2 - \nu^2,$$

$$R = \frac{c^2}{u^2} - l\lambda - m\mu - n\nu.$$

The condition (3) may now be satisfied by writing

$$f(t) = A + \frac{aB}{u}\log\left(1 + \frac{ut}{a}\right) + \frac{aB}{2u}\log\left(P\frac{u^2}{c^2}\right) \ldots\ldots (8),$$

$$T_{ab} = \frac{aB}{2u}\log\frac{R + (R^2 - PQ)^{\frac{1}{2}}}{R - (R^2 - PQ)^{\frac{1}{2}}} \ldots\ldots\ldots\ldots (9),$$

where A and B are arbitrary constants. Equation (4) then gives

$$\phi(\tau) = A + \frac{aB}{u}\log\left(1 + \frac{u\tau}{a}\right) + \frac{aB}{2u}\log\left(Q\frac{u^2}{c^2}\right) \ldots (10).$$

It is evident from the symmetry of this result that we can find a set of k observers $A_1, A_2, \ldots, A_k$ whose clocks are all together by specifying their motions as follows:

(A_p) $\quad x = l_p(a+ut),\ y = m_p(a+ut),\ z = n_p(a+ut) \ldots p = 1, 2, \ldots k.$

If $\quad P_p = \dfrac{c^2}{u^2} - l_p^2 - m_p^2 - n_p^2, \quad P_q = \dfrac{c^2}{u^2} - l_q^2 - m_q^2 - n_q^2,$

$$P_{pq} = \frac{c^2}{u^2} l_p l_q - m_p m_q - n_p n_q,$$

the clock belonging to the p^{th} observer should indicate at time t the number

$$f_p(t) = A + \frac{aB}{u}\log\left(1 + \frac{ut}{a}\right) + \frac{aB}{2u}\log\left(\frac{u^2}{c^2}P_p\right).$$

The interval between the p^{th} and q^{th} observers is then

$$T_{pq} = \frac{aB}{2u}\log\frac{P_{pq} + (P_{pq}^2 - P_p P_q)^{\frac{1}{2}}}{P_{pq} - (P_{pq}^2 - P_p P_q)^{\frac{1}{2}}}.$$

When $u \rightarrow 0$, $f_p(t)$ reduces to the form $A + Bt$ and T_{pq} becomes simply $(B/c)\, r_{pq}$, where r_{pq} is the distance between the two observers A_p, A_q.

A set of observers at constant intervals from one another who are provided with clocks which are all running together will be called an *organised set of observers.* It is clear from the above example that the different observers may or may not be at rest relatively to one another.

Reflexion in a moving plane mirror.

§ 3. The image of a point source (x, y, z, t) in a plane mirror moving, with uniform velocity v in a direction perpendicular to itself, may be obtained by means of the transformation*

$$\left.\begin{aligned} x' &= x - \frac{2c^2}{c^2 - v^2}(x - vt) \\ t' &= t - \frac{2v}{c^2 - v^2}(x - vt) \\ y' &= y, \quad z' = z \end{aligned}\right\} \qquad (11),$$

where $x = vt$ is the equation of the moving mirror and (x', y', z', t) the coordinates of the image. These equations may be obtained very easily by noticing that the locus of the points in which rays of light, issuing from the point x, y, z at time t, strike the moving mirror is a quadric of revolution having the source of light as one focus. The image of the source is at the other focus of the quadric, and it is easy to calculate the time at which light must leave the image in order to coincide with the rays from the source which have been reflected.

We shall say that the above equations give the image whether the velocity v is less than or greater than the velocity of light.

It is easy to see that the moving mirror is completely and uniquely determined when a point source and its image are given. To prove this we shall make use of a representation of a point source (x, y, z, t) by means of a directed sphere† of radius ct, whose centre is at the point (x, y, z).

In the first place it should be noticed that

$$\frac{c(t' - t)}{x' - x} = \frac{v}{c}.$$

Hence if θ is the semi-vertical angle of the tangent cone, whose vertex is at the centre of similitude of the two directed spheres representing the point source and its image, we have the relation

$$\sin\theta = v/c \qquad (12).$$

* H. Bateman, *Phil. Mag.* Dec. (1909), May (1910). V. Varicák, *Phys. Zeitschr*, Bd. xi. (1910), p. 586.

† H. Bateman. *Phil Mag* Oct. 1910). *Amer. Jour.* (1912). H. E. Timerding, *Jahrest d. Deutsch. Math. Verein*, Bd. 21 (1912). K. Ogura, *Science Reports*, Tôhoku Univ. Vol. II. (1913).

It is clear that the velocity of the mirror is greater than or less than that of light according as the centre of similitude is inside or outside the spheres.

The plane through the common points of the two spheres can be identified with the initial position of the mirror; for since the equations of the spheres are

$$(X-x)^2+(Y-y)^2+(Z-z)^2=c^2t^2,$$
$$(X-x')^2+(Y-y')^2+(Z-z')^2=c^2t'^2,$$

respectively, the plane through their common points is

$$(x'-x)(2X-x-x')+c^2(t'-t)(t'+t)=0.$$

Now $x+x'=v(t+t')$ and $c^2(t'-t)=v(x'-x)$, consequently the above equation reduces to $X=0$. The initial position and velocity of the mirror being known, its motion is completely determined.*

The determination of the time and position of an event from the recorded times at which it is witnessed by an organised set of observers.

§ 4. Consider an organised set of four observers A_1, A_2, A_3, A_4, whose clocks indicate the numbers T_1, T_2, T_3, T_4 respectively when the event is witnessed. If T is the required clock-time at which the event occurred, the quantities

$$T_1-T, \quad T_2-T, \quad T_3-T, \quad T_4-T$$

are the intervals between the event and the four observers. We shall denote these by the symbols T_{01}, T_{02}, T_{03}, T_{04} respectively, and the intervals between the different observers by the symbols T_{23}, T_{31}, T_{12}, T_{14}, T_{24}, T_{34} respectively. The problem is to express T in terms of T_1, T_2, T_3, T_4 by means of a relation of type

$$T=f(T_1,\ T_2,\ T_3,\ T_4) \qquad \text{.................(I.)}$$

If we consider the special case in which the event occurs at a point occupied by the observer A_1, we have, when $T=T_1$ and $T_2=T_1+T_{12}$, $T_3=T_1+T_{13}$, $T_4=T_1+T_{14}$; hence the function f must necessarily satisfy the functional equation

$$T_1=f(T_1,\ T_1+T_{12},\ T_1+T_{13},\ T_1+T_{14}).$$

* When we use Minkowski's representation of a space-time point x, y, z, t by a point with rectangular coordinates (x, y, z, ict) in a space of four dimensions S_4, a reflexion in a plane mirror moving with uniform velocity is represented by a reflexion in an hyperplane, and the theorem becomes obvious. This representation of a space-time point ought perhaps to be associated with the name of Poincaré. Cf. *Rend. Palermo*, t. 21 (1906), p. 168.

Similarly it can be shown that it must satisfy three other functional equations of a similar character.

The relation (I.) can be found very quickly when it is known that the ten mutual intervals

$$T_{01},\ T_{02},\ T_{03},\ T_{04},\ T_{23},\ T_{31},\ T_{12},\ T_{14},\ T_{24},\ T_{34}$$

are connected by an identical relation. It should be noticed, moreover, that since the last six quantities are constant, the fourth quantity can be regarded as a function of the other three, and is consequently constant when the other three quantities are constant. We may regard the first three intervals, or three independent functions of them, as coordinates fixing the position of the point at which the event occurred. If we call these coordinates X, Y, Z, the time T is given by the equation

$$T = T_4 - T_{04} = T_4 - \psi(X,\ Y,\ Z).$$

It is of course important that a good choice of coordinates X, Y, Z should be made. Let us consider two events which are witnessed by the observers A_1, A_2, A_3, A_4 at times T_1, T_2, T_3, T_4 and $T_1+\delta T_1$, $T_2+\delta T_2$, $T_3+\delta T_3$, $T_4+\delta T_4$ respectively, where δT_1, δT_2, δT_3, δT_4 are small quantities. Let us suppose, moreover, that the second event consists of a signal from an observer Q indicating that he has just witnessed the first event. When this is the case the four increments δT_1, δT_2, δT_3, δT_4 will not be independent, but will be connected by an identical relation which will be assumed to be of the form

$$\sum_{m,n} B_{m,n}\,\delta T_m \,.\, \delta T_n = 0, \qquad m,\ n = 1\,.\,2\,.\,3\,.\,4,$$

where $B_{m,n}$ is a function of T_1, T_2, T_3, T_4. This relation can be expressed in the form

$$A\delta X^2 + B\delta Y^2 + C\delta Z^2 + D\delta T^2 + 2F\delta Y\delta Z + 2G\delta Z\delta X$$
$$+ 2H\delta X\delta Y + 2U\delta X\delta T + 2V\delta Y\delta T + 2W\delta Z\delta T = 0,$$

where the coefficients are functions of X, Y, Z, T. Now since this quadratic equation is of fundamental importance in the description of the propagation of light* by means of the

* It can be regarded as the equation determining the form of the wave front of an elementary wave issuing from the point X, Y, Z at time T. An attempt to formulate a scheme of electromagnetic equations consistent with the above equation has been made by the author. *Proc. Lond. Math. Soc.*, ser. 2, vol. viii. (1910).

coordinates X, Y, Z, T it is natural to endeavour to choose X, Y and Z so that the above equation takes a simple form such as

$$A\,\delta X^2 + B\,\delta Y^2 + C\delta Z^2 + D\,\delta T^2 = 0.$$

Let us now consider a simple case in which this can be done. If the four observers are stationary relative to one another and space is Euclidean, and the configuration of the four observers A_1, A_2, A_3, A_4 is either at rest or is moving uniformly without rotation relative to our standard set of axes, we may assume that the ten quantities T_{pq} are proportional to the mutual distances of five points in space. They are consequently connected by the identical relation*

$$\begin{vmatrix} T_{01}^2 & T_{02}^2 & T_{03}^2 & T_{04}^2 & 1 & 0 \\ 1 & 1 & 1 & 1 & 0 & 1 \\ T_{41}^2 & T_{42}^2 & T_{43}^2 & 0 & 1 & T_{40}^2 \\ T_{31}^2 & T_{32}^2 & 0 & T_{34}^2 & 1 & T_{30}^2 \\ T_{21}^2 & 0 & T_{23}^2 & T_{24}^2 & 1 & T_{20}^2 \\ 0 & T_{12}^2 & T_{13}^2 & T_{14}^2 & 1 & T_{10}^2 \end{vmatrix} = 0 \ldots\ldots (\mathrm{II})$$

This is a quadratic equation for the determination of T. We may discuss it geometrically by considering the four directed spheres representing the points A_1, A_2, A_3, A_4, at times T_1, T_2, T_3, T_4, respectively, and a sphere S of radius cT whose centre is at the point where the event occurred. The equation then expresses that this last directed sphere touches the first.

Now there are two directed spheres S and S' which touch the four given directed spheres, and they are the representative spheres of two point sources which are images of one another in the moving plane mirror which passes through the points A_1, A_2, A_3, A_4, at times T_1, T_2, T_3, T_4, respectively. This follows at once from the fact that rays of light starting from the two point sources will either arrive at A_1, A_2, A_3, A_4, at times T_1, T_2, T_3, T_4, respectively, or can be supposed to have passed through these points at the respective times.

The plane containing the centres of similitude of each pair of directed spheres of the set of four, with centres at A_1, A_2, A_3, A_4, respectively, is the initial position of the mirror. The velocity of the mirror can be found from the

* Scott and Mathews "Theory of Determinants" (1904), p. 239. The relation is due to Cayley.

ratio of the radius of one of these spheres to the distance of its centre from the plane just mentioned. If the plane does not cut the spheres in real points the velocity of the mirror is greater than that of light.

If the velocity of the mirror is greater than that of light the centre of similitude of the two directed spheres S, S' lies within the two spheres, and it is easy to see that the radius of any directed sphere such as 'T_1', which touches both, is intermediate between the radii of S and S'. In this case there is only one value of T less than each of the quantities T_1, T_2, T_3, T_4 which satisfies our quadratic equation. The position and time of the event can then be determined uniquely, for when T is known the distances of the place of occurrence from A_1, A_2, A_3, A_4 are known and the ordinary rectangular coordinates of the place can be found without difficulty. On the other hand, if the velocity of the mirror is less than that of light, the centre of similitude of the two directed spheres S, S' lies outside the two spheres, and it is easy to see that the radius of a directed sphere which touches both is not intermediate between the radii of the two spheres S and S'. Hence in this case there are either two solutions of the problem or no solution at all.

Let us now consider an organised set of five observers whose mutual intervals are connected by the identical relation corresponding to that between the mutual distances of five points in space. If a value of T, calculated from the observations of one set of four observers, agrees with a value of T calculated from the observations of another set of four observers, all is well. If, however, the value of T calculated in the different ways do not agree, a reason must be found for it. Several possible causes of the disagreement may be suggested.

1. The observers may be moving relatively to one another.

2. Space may be non-Euclidean.

3. The observers may be at rest relatively to one another, but the configuration of four observers may not be moving uniformly relatively to our standard axes, and consequently the assumption that T_{pq} is proportional to the distance between A_p and A_q is unjustifiable.

Geometrical representation of the interval between two moving points in certain particular cases.

§ 5. In the motion considered in § 2 the two points A and B pass through the origin at the same time $t=-a/u$ and

travel along straight lines with constant velocities. If we adopt Poincaré's representation of a space-time point (x, y, z, t) by a point with rectangular coordinates (x, y, z, ict) in a space of four dimensions, the two moving points A and B will be represented by two intersecting straight lines whose direction cosines are proportional to $(l, m, n, ic/u)$ and $(\lambda, \mu, \nu, ic/u)$ respectively. If θ is the angle between these lines we have

$$\cos\theta = -\frac{R}{\sqrt{(PQ)}},$$

hence
$$T_{ab} = \pm \frac{iaB\theta}{u} = \pm ik\theta \text{ say.}$$

When we have an organised set of five points, which are moving according to the equations

$$x = l_p(a + ut), \quad y = m_p(a + ut), \quad z = n_p(a + ut), \quad p = 1.2\ldots5,$$

we may deduce an identical relation between the mutual intervals of the five points from the well-known relation between the mutual inclinations of five points in a space of four dimensions.* If $c_{rs} = \cosh(kT_{rs})$ the identical relation is

$$\begin{vmatrix} 1 & c_{12} & c_{13} & c_{14} & c_{15} \\ c_{21} & 1 & c_{23} & c_{24} & c_{25} \\ c_{31} & c_{32} & 1 & c_{34} & c_{35} \\ c_{41} & c_{42} & c_{43} & 1 & c_{45} \\ c_{51} & c_{52} & c_{53} & c_{54} & 1 \end{vmatrix} = 0 \ldots\ldots\ldots\ldots(\text{III.}).$$

This may be regarded as an equation for the determination of k. If now we have observers at the moving points who witness an event at the clock-times $T_1, T_2, \ldots, T_5$ respectively and T is the required clock-time for the event, the differences $T_1 - T, \; T_2 - T, \; T_3 - T, \; T_4 - T, \; T_5 - T$ will be the intervals between the event and the observers. With the aid of the identical relation of type (III.) between the intervals of five points we may deduce an equation for T from the observations of each set of four observers obtained by leaving out one of the five observers. The different values for T which are found in this way ought to agree; if they do not there may be several possible explanations of the discrepancy just as in § 4.

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* Scott and Mathews, *loc. cit.*

A more general type of motion, in which the interval between two moving points can be expressed in a simple form, and interpreted geometrically, may be obtained by transforming the uniform rectilinear motion considered in § 2 by means of a transformation of the coordinates (x, y, z, t) which leave the equation

$$(x-x_1)^2+(y-y_1)^2+(z-z_1)^2-c^2(t-t_1)^2=0$$

unaltered in form. Such a transformation corresponds to a conformal transformation in Minkowski's four-dimensional space S_4, and consequently makes two intersecting straight lines in S_4 correspond in general to two circles intersecting in two points. Either or both of the circles can degenerate into straight lines.

Since a conformal transformation in S_4 leaves the angle between two curves unaltered, it follows that the interval between the two moving points corresponding to two doubly intersecting circles is proportional to the angle between the two circles.

A circle in S_4 corresponds to a point moving along a conic in our space*. If we represent each position of the moving point by a directed sphere we obtain a chain of directed spheres with some notable properties.

To discuss the motion it will be sufficient to consider the transformation which corresponds to an inversion in S_4. This is specified analytically by the equations

$$x'=b^2\frac{x}{s^2},\quad y'=b^2\frac{y}{s^2},\quad z'=b^2\frac{z}{s^2},\quad t'=b^2\frac{t}{s^2},$$

where $s^2=x^2+y^2+z^2-c^2t^2$ and b is a constant.

It may be specified geometrically by saying that the representative spheres of two corresponding space-time points (x, y, z, t), (x', y', z', t') are transformed into one another by an ordinary inversion with respect to a sphere of radius b whose centre is at the origin.†

Let us now apply this transformation to a system of points moving along straight lines in the manner specified by the equations

$$x=\alpha+l_p(a+ut),\quad y=\beta+m_p(n+ut),\quad z=\gamma+n_p(a+ut),$$
$$p=1, 2, \ldots.$$

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* This type of motion has been considered by Born, *Ann. d. Phys.*, Bd. 30 (1909); Sommerfeld, *Ann. d. Phys.*, Bd. 33 (1912), p. 673; Kottler, *Weiner Berichte*, Bd. 121 (1912); Hassé. *Proc. Lond. Math. Soc.*, ser. 2, vol. xii. (1913), p. 181; Schott, *Electromagnetic Radiation* (1912), p. 63.

† H. Bateman, *Proc. Lond. Math. Soc.*, ser. 2, vol. vii. (1909), p. 84.

The representative spheres of one of these moving points have their centres on a straight line and have a common centre of similitude which lies within all the spheres if the velocity of the moving point is always less than that of light. It is easy to see that two spheres whose radii have the same sign do not intersect, also that there are two spheres through each point in space and therefore two through the centre of inversion. The inverse system of spheres must therefore contain two planes, and so the curve described by the moving point extends to infinity and is consequently an hyperbola.

Since the line through the centre of similitude and the centre of inversion cuts each member of the first set of spheres at the same angle, it also cuts each member of the second set of spheres at the same angle. The system of spheres obtained by inversion thus consists of spheres whose centres lie on an hyperbola and which cut a chord of the hyperbola at a fixed angle, which is the complement of half the angle between the asymptotes of the hyperbola; the chord is the major axis of the hyperbola. The motion may be specified analytically by substituting the above expressions for x, y, z in the previous equation.

Since a circle in S_4 cuts a space $t=\text{const.}$ in two points it follows that the circle really corresponds to the motion of two associated particles describing different branches of the same hyperbola. To obtain an organised system of observers moving along hyperbolic paths the representative circles in S_4 must have two points in common, consequently each hyperbolic motion and its associated motion must be such that either the moving particle or the associated moving particle passes through two fixed points at specified times, it is possible of course that one particle may pass through one of the fixed points and the associated particle through the other.

The cases that have been considered are clearly not the only cases in which a simple expression for the interval can be found. Let us suppose for instance that the observer A is at rest while B moves along a straight line through A according to the law $x=\chi(t)$, then t_1 and t_2 satisfy the equations

$$c(\tau-t_1)=\chi(\tau), \quad c(t_2-\tau)=\chi(\tau).$$

Hence the function $f(t)$ must satisfy the functional equation

$$f\left[\tau+\frac{1}{c}\chi(\tau)\right]-f\left[\tau-\frac{1}{c}\chi(\tau)\right]=\text{const.}=2T_{ab}.$$

If the function $f(t)$ is given it is generally easy to find the function $\chi(t)$, for instance if $f(t) = t^2$, we must have

$$\frac{4\tau}{c}\chi(\tau) = 2T_{ab}$$

or

$$\chi(\tau) = \frac{cT_{ab}}{2\tau}.$$

A geometrical representation of space-time vectors.

§ 6. The four-dimensional vector analysis introduced into the theory of relativity by Minkowski* admits of an interesting geometrical treatment with the aid of directed spheres.†

A 4-vector whose components are A_1, A_2, A_3, A_4 may, for instance, be represented by the relation to a sphere S, whose centre is at the origin and whose radius is C of another sphere whose centre is at the point A_1, A_2, A_3 and whose radius is $A_4 + c$. It is more convenient, however, to represent the 4-vector by the relation to the sphere S of a point P and an associated number ν. This point P is the centre of similitude of the two directed spheres just mentioned; its coordinates x, y, z are determined by the equations

$$\nu x = A_1, \quad \nu y = A_2, \quad \nu z = A_3, \quad -\nu c = A_4 \ldots\ldots (13).$$

If ν is regarded as analogous to a mass, the point and number representing the sum of a number of 4-vectors are found by determining the centre of mass of the masses at the different representative points of the vectors, and associating with it the sum of the masses. If the representative point P of a 4-vector lies within the sphere S, the vector is said to be *time-like*, if it lies outside the sphere, the vector is said to be *space-like*.

If (u, v, w) are the component velocities of a moving point, the four quantities $(u, v, w, -c)$ may be regarded as proportional to the components of a 4-vector which is time-like or space-like according as the velocity of the moving point is less than or greater than that of light. If the point moves with the velocity of light the representative point is on the sphere and the 4-vector is special.

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* *Gött. Nachr.* (1908). *Phys. Zeitschr.* (1909). See also G. N. Lewis, *Proc. Amer. Acad. of Arts and Sciences*, Oct (1910). A. Sommerfeld, *Ann. d. Phys.* Bd. 32 (1910), p. 765; Bd. 33 (1910), p. 651. L. Silberstein, *The Theory of Relativity* (1914). E. Cunningham, *The Principle of Relativity* (1914). B. Cabrera, *Revista d. R. Acad.* Madrid, t. 12 (1913), pp. 546, 738 E. B. Wilson and G. N. Lewis, *Proc. Amer. Acad. of Arts and Sciences*, vol xlviii. (1912).

† H. Bateman, *Phil. Mag.* Oct. (1910).

The angle between two 4-vectors can be defined in the following way. Let P and P' be the two representative points and let a circle be drawn to pass through P and P' and to cut the sphere S orthogonally. The angle subtended by the chord PP' at a point of this circle is then the angle between the two vectors. If P and P' are conjugate points with regard to the sphere S, they are at the extremities of a diameter of the circle and the angle is in this case a right angle.

Hence two perpendicular 4-vectors are represented by points which are conjugate with respect to S. A set of four mutually perpendicular 4-vectors are thus represented by the vertices of a tetrahedron which is self-polar with respect to S. It is clear that one, and only one, of the four vectors can be time-like.

If two 4-vectors (A_1, A_2, A_3, A_4), (B_1, B_2, B_3, B_4) are represented by numbers ν, μ at the points P, Q, respectively, the special 6-vector whose components are $A_2B_3 - A_3B_2$, $A_3B_1 - A_1B_3$, $A_1B_2 - A_2B_1$, $A_1B_4 - A_4B_1$, $A_2B_4 - A_4B_2$, $A_3B_4 - A_4B_3$, may be represented geometrically by a force* of magnitude $\mu\nu PQ$ acting in the direction PQ along the line PQ.

The two reciprocal 6-vectors whose components are $(F_{23}, F_{31}, F_{12}, F_{14}, F_{24}, F_{34})$, $(F_{14}, F_{24}, F_{34}, -F_{23}, F_{31}, -F_{12})$, respectively, are represented by forces acting along lines which are polar lines with regard to the sphere S, provided of course that the relation

$$F_{23}F_{14} + F_{31}F_{24} + F_{12}F_{34} = 0$$

is satisfied. When this relation is not satisfied, the 6-vector F cannot be represented by a single force. It can be represented by a wrench, but it is more convenient to represent it by means of two forces acting along lines which are polar lines with respect to the sphere S.

Let $(H_x, H_y, H_z, E_x, E_y, E_z)$ be the components of a general 6-vector, and let $(h_x, h_y, h_z, e_x, e_y, e_z)$ be the components of a special 6-vector which is the vector product of the two 4-vectors A and B.

Using the ordinary notation for vectors in a space of three

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* The first three components of the 6-vector are equal to the moments of the force about the axes, and the last three to the three components of the force multiplied by c. The sum of a number of special 6-vectors may be represented geometrically by a system of forces or its simplest equivalent obtained by the composition of forces.

dimensions we shall endeavour to find vectors $\mathbf{h}$ and $\mathbf{e}$ such that*

$$\mathbf{H} = \mathbf{h} + \kappa\mathbf{e} \quad \text{and} \quad \mathbf{E} = \mathbf{e} - \kappa\mathbf{h} \qquad \text{...........(14)},$$

whers κ is a scalar quantity. Since $(\mathbf{eh}) = 0$, the constant κ is determined by the equation

$$(1 - \kappa^2)(\mathbf{EH}) - \kappa(\mathbf{EE}) + \kappa(\mathbf{HH}) = 0 \qquad \text{.........(15)},$$

and the vectors $\mathbf{e}$, $\mathbf{h}$, may then be determined from the preceding equations.

Now let us consider the case when $\mathbf{E}$ and $\mathbf{H}$ are the electric and magnetic forces at an arbitrary point in an electro-magnetic field. We shall regard the 4-vectors A and B as proportional to velocity 4-vectors, so that their components are $(av_x, av_y, av_z, -ac)$ and $(bu_x, bu_y, bu_z, -bc)$ respectively. We then have a representation of the vectors $\mathbf{E}$ and $\mathbf{H}$ in terms of two velocities $\mathbf{u}$ and $\mathbf{v}$,

$$\left.\begin{aligned} \mathbf{H} &= ab[\mathbf{vu}] + c\kappa ab(\mathbf{u} - \mathbf{v}) \\ \mathbf{E} &= cab(\mathbf{u} - \mathbf{v}) - \kappa ab[\mathbf{vu}] \end{aligned}\right\} \qquad \text{............(16)}.$$

Similarly if the 6-vector $(e_x, e_y, e_z - h_x, h_y, h_z)$ is the vector product of two 4-vectors A' and B', whose components are $(a'v'_x, a'v'_y, a'v'_z, -a'c)$ and $(b'v'_x, b'v'_y, b'v'_z, -b'c)$ respectively, we have a second representation of $\mathbf{E}$ and $\mathbf{H}$, viz.

$$\left.\begin{aligned} \mathbf{H} &= -a'b'c(\mathbf{u}' - \mathbf{v}') + \kappa a'b'[\mathbf{v}'\mathbf{u}'] \\ \mathbf{E} &= a'b'[\mathbf{v}'\mathbf{u}'] + c\kappa a'b'(\mathbf{u}' - \mathbf{v}') \end{aligned}\right\} \qquad \text{.........(17)}.$$

The 6-vector $(\mathbf{H}, \mathbf{E})$ is thus represented as the sum of two special 6-vectors $(\mathbf{h}, \mathbf{e})$, $(\kappa\mathbf{e}, -\kappa\mathbf{h})$, and these can be represented geometrically by forces acting along lines L, L', which are polar lines with respect to the sphere S. The 4-vectors A and B are represented geometrically by numbers a and b associated with two points A and B on the line L, while the 4-vectors A' and B' are represented geometrically by numbers a' and b' associated with two points A' and B' on the line L'. Since one of the two lines, say L, cuts the sphere S in real points we can choose either one or both of the points A and B so that they lie within the sphere S, consequently we can choose the velocities $\mathbf{u}$ and $\mathbf{v}$ if necessary so that they are less than the velocity of light, but the velocities $\mathbf{u}'$, $\mathbf{v}'$ will be

* Cf. E. B. Wilson and G. N. Lewis, *Amer. Proc. Acad of Arts and Sciences*, vol. xlviii., Nov. (1912). H. Bateman, *Proc. Lond. Math. Soc.*, ser. 2, vol. x. (1911), p. 96.

greater than the velocity of light, except perhaps in the case when the lines L and L' touch the sphere S, one of the velocities $\mathbf{u}'$, $\mathbf{v}'$ can then be equal to the velocity of light.

We shall now show that the velocity $\mathbf{v}$ is a possible velocity for the aether in the electromagnetic field $(\mathbf{H}, \mathbf{E})$. To do this we must prove that $\mathbf{v}$ satisfies Cunningham's relation*

$$c^2\mathbf{g} + \mathbf{E}(\mathbf{vE}) + \mathbf{H}(\mathbf{vH}) = \mathbf{v}\{2w - (\mathbf{vg})\} \ldots\ldots (18),$$

where $\mathbf{g} = [\mathbf{EH}]/c$ and $w = \frac{1}{2}(\mathbf{EE}) + \frac{1}{2}(\mathbf{HH})$.

Now the equations (16) give

$$c\mathbf{g} = [\mathbf{EH}] = ca^2b^2(1+\kappa^2)\{[\mathbf{u}(\mathbf{vu})] - [\mathbf{v}(\mathbf{vu})]\}.$$

Now
$$[\mathbf{u}(\mathbf{vu})] = \mathbf{v}(\mathbf{uu}) - \mathbf{u}(\mathbf{uv})$$
$$[\mathbf{v}(\mathbf{vu})] = \mathbf{v}(\mathbf{uv}) - \mathbf{u}(\mathbf{vv}).$$
therefore
$$\mathbf{g} = a^2b^2(1+\kappa^2)\{\mathbf{v}[(\mathbf{uu}) - (\mathbf{uv})] + \mathbf{u}[(\mathbf{vv}) - (\mathbf{uv})]\}.$$
Again
$$(\mathbf{vE}) = cab\{(\mathbf{uv}) - (\mathbf{vv})\}, \quad (\mathbf{vH}) = c\kappa ab\{(\mathbf{uv}) - (\mathbf{vv})\},$$
therefore
$$\kappa(\mathbf{vE}) = (\mathbf{vH}).$$
Also $\mathbf{E}(\mathbf{vE}) + \mathbf{H}(\mathbf{vH}) = c^2a^2b^2(1+\kappa^2)\{(\mathbf{uv}) - (\mathbf{vv})\}\{\mathbf{u} - \mathbf{v}\}$,
consequently
$$c^2\mathbf{g} + \mathbf{E}(\mathbf{vE}) + \mathbf{H}(\mathbf{vH}) = c^2a^2b^2(1+\kappa^2)\mathbf{v}\{(\mathbf{uu}) + (\mathbf{vv}) - 2(\mathbf{uv})\}.$$
On the other hand
$$2w = (\mathbf{EE}) + (\mathbf{HH}) = a^2b^2(1+\kappa^2)[\{(\mathbf{uu})(\mathbf{vv}) - (\mathbf{uv})^2\} + c^2\{(\mathbf{uu}) + (\mathbf{vv}) - 2(\mathbf{uv})\}]$$
and
$$(\mathbf{vg}) = a^2b^2(1+\kappa^2)\{(\mathbf{uu})(\mathbf{vv}) - (\mathbf{uv})^2\},$$
therefore $a^2b^2c^2(1+\kappa^2)\{(\mathbf{uu}) + (\mathbf{vv}) - 2(\mathbf{uv})\} = 2w - (\mathbf{vg})$.

The relation (18) is now established. In a similar way it can be proved that the velocities u, u', v' are possible velocities of the aether. It should be noticed that since the lines L and L' are polar lines with regard to the sphere S we have the relations

$$(\mathbf{uu}') = c^2, \quad (\mathbf{uv}') = c^2, \quad (\mathbf{vu}') = c^2, \quad (\mathbf{vv}') = c^2.$$

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* *Proc. Roy. Soc.* vol. lxxxiii. (1909), p. 110. *The Principle of Relativity*, ch. xv.

Five years ago Mr. Cunningham remarked to me in a letter that my vector* s, which satisfies the relations

$$\left.\begin{array}{c} c^2\mathbf{E} + c[\mathbf{sH}] = \mathbf{s}(\mathbf{sE}), \quad c^2\mathbf{H} - c[\mathbf{sE}] = \mathbf{s}(\mathbf{sH}) \\ (\mathbf{ss}) = c^2 \end{array}\right\} \ldots(19),$$

is a particular case of his vector $\mathbf{v}$. This may be proved as follows:—

We easily find from the above equations that

$$c^2[\mathbf{EH}] + c\mathbf{H}(\mathbf{sH}) - c\mathbf{s}(\mathbf{HH}) = [\mathbf{sH}](\mathbf{sE}),$$

or $$c^4\mathbf{g} + c^2\mathbf{E}(\mathbf{sE}) + c^2\mathbf{H}(\mathbf{sH}) = \mathbf{s}\{(\mathbf{sE})^2 + c^2(\mathbf{HH})\}.$$

Equations (19) also give

$$c^2(\mathbf{EE}) - c^3(\mathbf{sg}) = (\mathbf{sE})^2, \quad c^2(\mathbf{HH}) - c^3(\mathbf{sg}) = (\mathbf{sH})^2,$$

hence $$(\mathbf{sE})^2 + c^2(\mathbf{HH}) = c^2\{2w - (\mathbf{sg})\},$$

and so Cunningham's statement that $\mathbf{s}$ is a particular case of $\mathbf{v}$ is verified.

There are four possible vectors of type $\mathbf{s}$, and these are represented geometrically by the two pairs of points in which the two lines L and L' cut the sphere S.

If we compare the two representations (16) and (17) we have the relations

$$ab[\mathbf{vu}] = -a'b'c(\mathbf{u}' - \mathbf{v}'), \quad cab(\mathbf{u} - \mathbf{v}) = a'b'[\mathbf{v}'\mathbf{u}'] \ldots(20).$$

Now let us suppose that the components of the 4-vectors A, B, A', B' are proportional respectively to the partial derivatives of four functions α, β, α', β' with respect to x, y, z, ct, then the above relations take the form

$$\lambda \frac{\partial(\alpha, \beta)}{\partial(y, z)} = \frac{\mu}{c} \frac{\partial(\alpha', \beta')}{\partial(x, t)}, \quad \frac{\lambda}{c} \frac{\partial(\alpha, \beta)}{\partial(x, t)} = -\mu \frac{\partial(\alpha', \beta')}{\partial(y, z)} \ldots (21),$$

and we have the following representations† of $\mathbf{E}$ and $\mathbf{H}$,

$$\left.\begin{array}{l} H_x = \lambda \dfrac{\partial(\alpha, \beta)}{\partial(y, z)} - \mu\kappa \dfrac{\partial(\alpha', \beta')}{\partial(y, z)} = \dfrac{\mu}{c} \dfrac{\partial(\alpha', \beta')}{\partial(x, t)} + \dfrac{\kappa\lambda}{c} \dfrac{\partial(\alpha, \beta)}{\partial(x, t)} \\ E_x = -\kappa \dfrac{\mu}{c} \dfrac{\partial(\alpha', \beta')}{\partial(x, t)} + \dfrac{\lambda}{c} \dfrac{\partial(\alpha, \beta)}{\partial(x, t)} = -\mu \dfrac{\partial(\alpha', \beta')}{\partial(y, z)} - \kappa\lambda \dfrac{\partial(\alpha, \beta)}{\partial(y, z)} \end{array}\right\} \ldots(22),$$

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* *Phil. Mag.* Oct. (1910). *Proc. Lond. Math. Soc.* ser. 2, vol. viii. (1910), p. 469; vol. x. (1911), pp. 7, 96.

† These formulæ are a little more general than those given in a previous paper, *Proc. Lond. Math. Soc.*, ser. 2, vol. x. (1911), p. 96. It has not yet been proved that the above representation is possible whenever κ is constant.

The vectors **E** and **H** will certainly satisfy Maxwell's equations if λ is a function of α and β, $\mu\kappa$ a function of α' and β', and κ a constant. If these conditions are satisfied we may replace α and β by functions of these quantities in such a way as to make λ unity, and similarly we can make $\mu\kappa$ unity by a proper choice of the variables α', β'. If κ is not a constant there will be a volume density of electricity and convection currents in the field specified by (22).

The equations (21) may also be written in the following form,

$$\lambda \frac{\partial (x, t)}{\partial (\alpha', \beta')} = \frac{\mu}{c} \frac{\partial (y, z)}{\partial (\alpha, \beta)}, \quad \frac{\lambda}{c} \frac{\partial (y, z)}{\partial (\alpha', \beta')} = -\mu \frac{\partial (x, t)}{\partial (\alpha, \beta)} \text{......}(23).$$

If now we use the notation

$$\{\alpha\alpha'\} \equiv \frac{\partial x}{\partial \alpha}\frac{\partial x}{\partial \alpha'} + \frac{\partial y}{\partial \alpha}\frac{\partial y}{\partial \alpha'} + \frac{\partial z}{\partial \alpha}\frac{\partial z}{\partial \alpha'} - c^2 \frac{\partial t}{\partial \alpha}\frac{\partial t}{\partial \alpha'},$$

we may deduce from the preceding equations that

$$\{\alpha\alpha'\} = \{\alpha\beta'\} = \{\beta\alpha'\} = \{\beta\beta'\} = 0,$$

$$\lambda^2 [\{\alpha'\alpha'\}\{\beta'\beta'\} - \{\alpha'\beta'\}^2] = \mu^2 [\{\alpha\alpha\}\{\beta\beta\} - \{\alpha\beta\}^2].$$

Hence there is a relation of type

$$dx^2 + dy^2 + dz^2 - c^2 dt^2$$
$$= A dx^2 + 2H d\alpha d\beta + B d\beta^2 + A' d\alpha'^2 + 2H' d\alpha' d\beta' + B' d\beta'^2 \text{...}(24),$$

where A, H, B, A', H', B' are functions of α, β, α', β', satisfying the equation

$$\lambda^2 [A'B' - H'^2] = \mu^2 [AB - H^2] \text{..........}(25).$$

From a remark made above we may conclude that it is sufficient to put $\lambda = \mu = 1$ when endeavouring to determine functions α, β, α' and β', which satisfy the equations (24) and (25).

The case in which $(\mathbf{EH}) = 0$ is of special interest, for then $\kappa = 0$. The equations $\alpha =$ const, $\beta =$ const then give a moving line of magnetic force and the equations $\alpha' =$ const, $\beta' =$ const a moving line of electric force. It is frequently easy to find the functions α and β with the aid of the scalar and vector potentials.* To determine α' and β' we may endeavour

* See, for instance, the formulæ found by Hargreaves for the case of a moving point charge. These formulæ are given on p. 117 of the author's *Electrical and Optical Wave Motion*.

to choose A, H, and B so that the quadratic differential form

$$dx^2+dy^2+dz^2-c^2dt^2-(A\,d\alpha^2+2H\,d\alpha\,d\beta+B\,d\beta^2)$$

can be expressed in terms of the differentials of only two variables α' and β'. The functions α' and β' are both solutions of the equations

$$\frac{\partial\alpha}{\partial x}\frac{\partial\theta}{\partial x}+\frac{\partial\alpha}{\partial y}\frac{\partial\theta}{\partial y}+\frac{\partial\alpha}{\partial z}\frac{\partial\theta}{\partial z}=\frac{1}{c^2}\frac{\partial\alpha}{\partial t}\frac{\partial\theta}{\partial t},$$

$$\frac{\partial\beta}{\partial x}\frac{\partial\theta}{\partial x}+\frac{\partial\beta}{\partial y}\frac{\partial\theta}{\partial y}-\frac{\partial\beta}{\partial z}\frac{\partial\theta}{\partial z}=\frac{1}{c^2}\frac{\partial\beta}{\partial t}\frac{\partial\theta}{\partial t},$$

for θ. This method is successful in the case of the field due to a moving point charge and the functions found for α' and β' are derivable from the solutions of two Riccatian equations in accordance with a known result.*

Conjugate Electromagnetic Fields.

§ 7. Two fields $(\mathbf{E}, \mathbf{H})$, $(\mathbf{E}', \mathbf{H}')$ are said to be conjugate when the relations

$$(\mathbf{EH}')+(\mathbf{E}'\mathbf{H})=0, \quad (\mathbf{EE}')-(\mathbf{HH}')=0,$$

are satisfied. If now we represent each field by means of special 6-vectors (e, h) and (e', h') so that

$$\mathbf{H}=\mathbf{h}+\kappa\mathbf{e}, \qquad \mathbf{E}=\mathbf{e}-\kappa\mathbf{h},$$
$$\mathbf{H}'=\mathbf{h}'+\kappa'\mathbf{e}', \qquad \mathbf{E}'=\mathbf{e}'-\kappa'\mathbf{h}',$$

the above relations take the form

$$(1-\kappa\kappa')[(\mathbf{eh}')+(\mathbf{e}'\mathbf{h})]+(\kappa+\kappa')[(\mathbf{ee}')-(\mathbf{hh}')]=0,$$
$$(1-\kappa\kappa')[(\mathbf{ee}')-(\mathbf{hh}')]-(\kappa+\kappa')[(\mathbf{eh}')+(\mathbf{e}'\mathbf{h})]=0.$$

The determinant for these two linear equations is

$$(1-\kappa\kappa')^2+(\kappa+\kappa')^2=(1+\kappa^2)(1+\kappa'^2),$$

and this cannot vanish if the two fields are real, consequently we must have

$$(\mathbf{eh}')+(\mathbf{e}'\mathbf{h})=0, \quad (\mathbf{ee}')-(\mathbf{hh}')=0.$$

This means that the field $(\mathbf{e}, \mathbf{h})$ is conjugate to the field $(\mathbf{e}', \mathbf{h}')$.

Now let the special 6-vector $(\mathbf{e}, \mathbf{h})$ be represented by a force acting along a line L as in § 6, and the special 6-vector

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* *Amer. Journ. of Math.*, March (1915).

$(\mathbf{e}', \mathbf{h}')$ by a force acting along a line M. The condition $(\mathbf{e}\mathbf{h}') + (\mathbf{e}'\mathbf{h}) = 0$ then implies that the two lines L and M intersect, whilst the condition $(\mathbf{e}\mathbf{e}') - (\mathbf{h}\mathbf{h}') = 0$ implies that L intersects the polar line of M or that M intersects the polar line of L. Denoting by (v_x, v_y, v_z) the coordinates of the first point, and by (u_x, u_y, u_z) the coordinates of the point in which L intersects the polar line of M, we see that the field vectors in two conjugate fields can be expressed as follows:—

$$H = ab\,[\mathbf{vu}] + c\kappa ab\,(\mathbf{u}-\mathbf{v}) \quad = \kappa\alpha\beta\,[\mathbf{ww}'] - c\alpha\beta\,(\mathbf{w}'-\mathbf{w}),$$

$$E = cab\,(\mathbf{u}-\mathbf{v}) - \kappa ab\,[\mathbf{vu}] \quad = \alpha\beta\,[\mathbf{ww}'] + c\kappa\alpha\beta\,(\mathbf{w}'-\mathbf{w}),$$

$$H' = a'b'\,[\mathbf{vw}] + c\kappa' a'b'(\mathbf{w}-\mathbf{v}) \quad = \kappa'\alpha'\beta'\,[\mathbf{uu}'] - c\alpha'\beta'\,(\mathbf{u}'-\mathbf{u}),$$

$$E' = ca'b'\,(\mathbf{w}-\mathbf{v}) - \kappa' a'b'\,[\mathbf{vw}] = \alpha'\beta'\,[\mathbf{uu}'] + c\kappa'\alpha'\beta'\,(\mathbf{u}'-\mathbf{u}).$$

The point of chief importance is that the velocity of the aether $\mathbf{v}$ *can be the same for both fields.*

Conversely, if the velocity $\mathbf{v}$ of the aether is the same for two fields the two fields are not necessarily conjugate.

The above equations may be made more symmetrical by taking $\mathbf{u}' = \mathbf{w}'$. This is permissible since L and its polar both meet M and its polar.

A CONDITION FOR THE VALIDITY OF TAYLOR'S EXPANSION.

By *T. W. Chaundy*, Christ Church, Oxford.

THE conditions for the validity of Taylor's expansion of a function of a real variable have been given by Pringsheim* and W. H. Young.†

Pringsheim proves that the necessary and sufficient condition that $\sum_0^\infty \frac{x^n f_n(a)}{n!}$ should converge to $f(a+x)$ over the interval $0 \leq x < R$ is that $\frac{f_n(a+x)y^{n+p}}{(n+p)!}$, as n tends to infinity, should tend to zero, uniformly for all values of x, y for which $0 \leq x \leq x+y \leq r$, where $r < R$ and p is some convenient integer.

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* *Math. Annalen*, vol. xliv., p. 57.

† *Quar. Jour. of Math.*, vol. xl., p. 157; "The fundamental theorems of the differential calculus" (Camb. 1910), p. 57.

W. H. Young gives the simpler condition that for each fixed positive $r < R$ the function

$$\left| \frac{(R-r)^{n-p}}{(n-p)!} f_n(a+x) \right|,$$

regarded as a function of the two variables (x, n), should be bounded in the region $n \geq p$, $0 < x \leq R$.

I seek to establish the following results:

(1) *If $f(a+x)$ can be expanded in a power series $\sum_0^\infty A_n x^n$ in the interval $0 \leq x \leq k$, and if the interval of convergence of this series is that given by $|x| < \rho$, then the function*

$$\left| \frac{f_n(a+x)}{n!} \right|^{\frac{1}{n}},$$

regarded as a function of the two variables x, n, is bounded in the region $n > 0$, $0 \leq x < k'$, where $k' < \rho$ and $\leq k$.

(2) *If*

$$\left| \frac{f_n(a+x)}{n!} \right|^{\frac{1}{n}}$$

is bounded in the region $n > 0$, $0 \leq x \leq k'$, and if

$$\sum_0^\infty \frac{x^n f_n(a)}{n!}$$

converges for $|x| < \rho$, the series will converge to $f(a+x)$ in the interval $0 \leq x \leq k$, where $k < \rho$, and $\leq k'$.

Similar results of course may be obtained for negative values of x.

If we are not concerned to be precise as to the end-points of these intervals, we may say briefly that the region of validity of the expansion of $f(a+x)$ in powers of x is the common territory of the region of convergence of $\sum \dfrac{x^n f_n(a)}{n!}$ and the region in which $\left| \dfrac{f_n(a+x)}{n!} \right|^{\frac{1}{n}}$ is bounded.

1. We have that

$$f(a+x) = A_0 + A_1 x + A_2 x^2 + \ldots \qquad \text{for } 0 \leq x \leq k,$$

and that the series converges for $0 \leq x < \rho$.

Being given any $k' < \rho$ and $\leq k$, choose ρ' such that $k' < \rho' < \rho$. Then since $\overline{\lim}. |A_n|^{\frac{1}{n}} = \rho^{-1}$, we have that

$$|A_n|^{\frac{1}{n}} < \rho'^{-1}, \text{ if } n > \text{some } N.$$

In the interval $0 \leq x \leq k'$, since $k' < \rho$, we may differentiate the series n times and obtain

$$\frac{f_n(a+x)}{n!} = A_n + (n+1)A_{n+1}x + \frac{(n+1)(n+2)}{2!}A_{n+2}x^2 + \dots.$$

Thus

$$\left|\frac{f_n(a+x)}{n!}\right| < \frac{1}{\rho'^n}\left\{1+(n+1)\frac{k'}{\rho'} + \frac{(n+1)(n+2)}{2!}\frac{k'^2}{\rho'^2} + \dots\right\},$$

$$\text{if } n > N, \text{ i.e. } < (\rho' - k')^{-n-1}$$
$$< (\rho' - k')^{-2n}.$$

Hence $$\left|\frac{f_n(a+x)}{n!}\right|^{\frac{1}{n}} < (\rho' - k')^{-2}, \text{ for every } n > N,$$

and every x in the interval $0 \leq x \leq k'$.

But, for a fixed s, $f_s(a+x)$ is bounded in the interval $0 \leq x \leq k'$; the same is therefore true of

$$\left|\frac{f_s(a+x)}{s!}\right|^{\frac{1}{s}},$$

and also of the aggregate of these functions when $s = 1, 2, 3, \dots, N$. It follows that we may remove the restriction $n > N$ in the preceding result, and say that

$$\left|\frac{f_n(a+x)}{n!}\right|^{\frac{1}{n}}$$

is bounded in the region

$$n > 0 \text{ and } 0 \leq x \leq k'.$$

2. We suppose that

$$\left|\frac{f_n(a+x)}{n!}\right|^{\frac{1}{n}} < M, \text{ when } 0 \leq x \leq k'.$$

Now $f(a+x) = f(a) + xf'(a) + \dots + \dfrac{x^n f_n(\theta x)}{n!}$, when $0 \leq \theta \leq 1$.

But if $0 \leq x \leq k'$, then $0 \leq \theta x \leq k'$, so that $\dfrac{f_n(\theta x)}{n!} < M^n$.

It is therefore sufficient to take $|x| < M^{-1}$ to secure that

$$\frac{x^n f_n(\theta x)}{n!} \to 0, \text{ as } n \to \infty.$$

In other words, $f(a+x)$ is equal to its Taylor-expansion in the interval $0 \leq x \leq K$ where $K \leq k'$, and $< M^{-1}$.

Now M is the upper bound (or at any rate a superior limit) to

$$\left|\frac{f_n(a+x)}{n!}\right|^{\frac{1}{n}}$$

in the interval $0 \leq x \leq k'$, while ρ^{-1} is the upper limit of

$$\left|\frac{f_n(a)}{n!}\right|^{\frac{1}{n}},$$

i.e. of the preceding function when $x=0$. Thus $M \geq \rho^{-1}$.

If $\rho = M^{-1}$ or if $k' \leq M^{-1}$, the restriction $K \leq k'$ and $< M^{-1}$ is identical with the restriction $K \leq k'$ and $< \rho$.

In this case we have established the validity of the expansion over the required interval. But if $M^{-1} < \rho$, and $< k'$, we have established the expansion over too small an interval. It will be shown that the restriction $K \leq k'$ and $< M^{-1}$ may be replaced by the restriction $K \leq k'$ and $< \rho$ by a process of "continuation."

Choose two positive quantities x, y, each $< 1/M$ and such that $x+y \leq k'$ and $< \rho$. Thus x, y are separately $\leq k'$ and $< \rho$, but $x+y$ need not be less than $1/M$ (if $1/M < k'$ and $< \rho$): it need only be less than $2/M$.

By Taylor's formula

$$f(x+y+a) = f(x+a) + yf'(x+a) + \dots$$
$$+ \frac{y^n f_n(x+a+\theta y)}{n!}, \text{ where } 0 \leq \theta \leq 1.$$

Since $|x+y| \leq k'$ so also $|x+\theta y| \leq k'$, and accordingly

$$\left|\frac{f_n(x+a+\theta y)}{n!}\right| < M^n.$$

But $y < 1/M$.

Hence
$$\frac{y^n f_n(x+a+\theta y)}{n!} \to 0 \text{ as } n \to \infty.$$

The expansion

$$f(x+y+a) = f(x+a) + yf'(x+a) + \frac{y^2}{2!} f''(x+a) + \dots$$

is therefore established for the specified range of values of x, y. But since $0 \leq x \leq k'$ and $x < M^{-1}$ we may expand $f(x+a)$, and therefore also $f'(x+a)$, $f''(x+a)$, ..., in powers of x. We then have $f(x+y+a)$ represented by a double series in powers of x, y. Summing "diagonally," *i.e.* first collecting terms of like degree in x, y, we have

$$f(x+y+a) = f(a) + (x+y)f'(a) + \frac{(x+y)^2}{2!} f''(a) + \dots.$$

Now this change in the order of summation is permissible, if the double series is absolutely convergent, *i.e.* if

$$\Sigma\Sigma \frac{x^m y^n}{m!\, n!} \, |f_{m+n}(a)|$$

is convergent.

This, being a series of positive terms, is convergent if

$$\Sigma \frac{(x+y)^p}{p!} \, |f_p(a)|$$

is convergent, *i.e.* if $\quad \Sigma \dfrac{(x+y)^p}{p!} f_p(a)$

is absolutely convergent. But, being a power series, it is absolutely convergent within its limits of convergence, *i.e.* if $x+y<\rho$, which has been stipulated.

Thus the expansion of $f(x+y+a)$ in powers of $(x+y)$ has been effected under the restriction $0 \leq x+y \leq K$, where $K<\rho$, $2/M$, and $\leq k'$.

We have thus replaced the conditions $K<1/M$ and $\leq k'$, where $(0, K)$ is the interval of validity of the expansion of $f(a+x)$ in powers of x, by the condition $K<2/M$, $<\rho$ and $\leq k'$. By a similar process of "continuation" we could replace these by $K<4/M$, $<\rho$ and $\leq k'$, and so on. It is clear then that ultimately we shall get $2^s/M>\rho$ or $>k'$, when this condition may be removed and we are left with $K<\rho$ and $\leq k'$, which is what we require.

The conditions of expansibility that have been proved establish the expansions of the elementary functions without difficulty, and apply also to the discussion of the expansion of a function of a function. Moreover, they "explain" why the expansion fails for such a function as e^{-1/x^2}, for then $f_n(x)$, in the neighbourhood of $x=0$, is of the order $e^{-1/x^2} x^{-3n}$, so that

$$\left| \frac{f_n(x)}{n!} \right|^{\frac{1}{n}}$$

is of the order $e^{-1/nx^2} x^{-3} (n!)^{-1/n}$. If we set $x^2=1/n$, this expression $\rightarrow \infty$ as $n \rightarrow \infty$ and $x \rightarrow 0$, so that

$$\left| \frac{f_n(n)}{n!} \right|^{\frac{1}{n}}$$

is not bounded in any region $0 \leq x \leq k$, $n>0$.

NOTE ON THE PRIMARY MINORS OF A CIRCULANT HAVING A VANISHING SUM OF ELEMENTS.

By *Sir Thomas Muir, LL.D.*

1. It is known from Borchardt that any axi-symmetric determinant having the sum of every row equal to zero has all its primary minors equal. Such a unique minor can be expressed in a variety of ways as a function of $\frac{1}{2}n(n-1)$ elements, the result being neatest and most convenient when the elements chosen are those outside the diagonal. Thus, if the given determinant be

$$\begin{vmatrix} a & b & c & d \\ b & e & f & g \\ c & f & h & i \\ d & g & i & j \end{vmatrix},$$

it is best that the letters removed with the help of the conditioning equations

$$\begin{aligned} 0 &= a+b+c+d, \\ &= b+e+f+g, \\ &= c+f+h+i, \\ &= d+g+i+j, \end{aligned}$$

be

$$a,\ e,\ h,\ j;$$

the minor in question, U_3 say, then being

$$\begin{vmatrix} b+c+d & -b & -c \\ -b & b+f+g & -f \\ -c & -f & c+f+i \end{vmatrix},$$

and its expansion*

$$dgi + dg(c+f) + di(f+g) + ig(b+c) \\ + (d+g+i)(bc+bf+cf).$$

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* It is worth noting that this expansion consists of three-letter combinations formed from b, c, d, f, g, i; and contains every such combination except

$$fgi,\ cdi,\ bdg,\ bcf,$$

each of these four being formed of the letters left on striking out a frame-line of the Pfaffian

$$\begin{vmatrix} b & c & d \\ & f & g \\ & & i \end{vmatrix}.$$

The case where $f, g, i = d, c, b$ is also worth noting.

Determinants of this type received at an early date the attention of Sylvester, who noted that the signs of all the terms in the final development are positive, and that the number of these terms in the case of the n^{th} order is

$$(n+1)^{n-1}.$$

2. Since a circulant is viewable as a special axi-symmetric determinant, it also must have a unique primary minor when the sum of its elements vanishes. On inquiry, however, the special is found to be not included in the general. The reason for this is that in the one case it is the diagonal of symmetry from which the elements have to be removed by substitution, and in the other it is not. Thus, the circulant being

$$\begin{vmatrix} l & m & n & r \\ r & l & m & n \\ n & r & l & m \\ m & n & r & l \end{vmatrix},$$

it is impossible to get rid of one of the letters by using the equation of condition

$$l+m+n+r=0$$

on the elements lying in the diagonal of symmetry, as was done in the previous instance; the substitution of $-m-n-r$ for l must now be made in the other diagonal, with the result that the unique minor, V_3 say, is seen to be

$$\begin{vmatrix} m+n+r & -m & -n \\ -r & m+n+r & -m \\ -n & -r & m+n+r \end{vmatrix},$$

the expansion of which is

$$m^3+2m^2n+n^2r+2n^2m+2n^2r+mr^2+2nr^2+4mnr+r^3.$$

3. In regard to the outward form of this determinant note should be taken that reading the elements of its three rows in succession we are merely repeating

$$m+n+r,\ \ -m,\ \ -n,\ \ -r,$$

$2\frac{1}{4}$ times as it were.

Similarly in the case of the next order, to obtain the sixteen required elements, we repeat

$$m+n+r+s,\ \ -m,\ \ -n,\ \ -r,\ \ -s,$$

$3\frac{1}{5}$ times.

This observation involves the fact that the determinant can be viewed as persymmetric; thus the one of the third order is

$$P(n,\ m,\ -m-n-r,\ r,\ n).$$

Further, either way of writing shows that the determinant is invariant to the interchange of m and r.

4. Closely related in form to the two determinants above is a third, of which a three-line example, W_3 say, is

$$\begin{vmatrix} u+v+w & -v & -w \\ -w & u+v+w & -v \\ -v & -w & u+v+w \end{vmatrix},$$

having the expansion

$$u^3+3u^2(v+w)+3u(v^2+vw+w^2),$$

with u in every term in accordance with the fact that each row-sum is equal to u.

All three determinants, U, V, W, are unisignants, the non-diagonal elements in every case being negative and yet all the row-sums positive. They differ in that the first is symmetric with respect to the main diagonal, the second persymmetric with respect to the secondary diagonal, and and the third circulant.

5. The third, W, being circulant is resolvable into linear factors; and therefore if it be expanded in descending powers of u, the last term of the expansion must be so resolvable. But the said last term can be shown to be, save for an arithmetical factor, a determinant of the form V. Consequently we have the important proposition that *the unique primary minor of a circulant having a vanishing sum of elements is resolvable into linear factors.*

The form of the factors will appear from the consideration of an individual case.

6. Taking W_5, with S written for $u+v+w+x+y$, and expanding it in Cayley's manner according to descending powers of a letter in the diagonal, we have

$$\begin{vmatrix} S & -v & -w & -x & -y \\ -y & S & -v & -w & -x \\ -x & -y & S & -v & -w \\ -w & -x & -y & S & -v \\ -v & -w & -x & -y & S \end{vmatrix}$$

$$= u^5 + 5u^4(v+w+x+y) + \ldots + 5u \begin{vmatrix} S-u & -v & -w & -x \\ -y & S-u & -v & -w \\ -x & -y & S-u & -v \\ -w & -x & -y & S-u \end{vmatrix}.$$

From this it follows that

$$\left\{\frac{W(u,\, v,\, w,\, x,\, y)}{u}\right\}_{u=0} = 5\,V(v,\, w,\, x,\, y).$$

But the W here being the circulant

$$C(S,\, -v,\, -w,\, -x,\, -y)$$

is expressible in the form

$$\begin{aligned} &(S - v\epsilon - w\epsilon^2 - x\epsilon^3 - y\epsilon^4)\cdot(S - v\epsilon^2 - w\epsilon^4 - x\epsilon - y\epsilon^3) \\ \cdot\,&(S - v\epsilon^3 - w\epsilon - x\epsilon^4 - y\epsilon^2)\cdot(S - v\epsilon^4 - w\epsilon^3 - x\epsilon^2 - y\epsilon) \\ \cdot\,&(S - v - w - x - y), \end{aligned}$$

where the last of the five factors is manifestly u, and ϵ is a prime fifth-root of unity. On dividing by u and thereafter putting $u = 0$ we thus have

$$\begin{aligned} &\{(1-\epsilon)\,v + (1-\epsilon^2)\,w + (1-\epsilon^3)\,x + (1-\epsilon^4)\,y\} \\ \cdot\,&\{(1-\epsilon^2)\,v + (1-\epsilon^4)\,w + (1-\epsilon)\,x + (1-\epsilon^3)\,y\} \\ \cdot\,&\{(1-\epsilon^3)\,v + (1-\epsilon)\,w + (1-\epsilon^4)\,x + (1-\epsilon^2)\,y\} \\ \cdot\,&\{(1-\epsilon^4)\,v + (1-\epsilon^3)\,w + (1-\epsilon^2)\,x + (1-\epsilon)\,y\} = 5\,V(v, w, x, y). \end{aligned}$$

Similarly, as a test of the extension in § 2, we have

$$\begin{aligned} 4\,V(m,\, n,\, r) &= [\{1 - \surd(-1)\}\,m + \{1+1\}\,n + \{1 + \surd(-1)\}\,r] \\ &\quad\cdot[\{1+1\}\,m + \{1-1\}\,n + \{1+1\}\,r] \\ &\quad\cdot[\{1 + \surd(-1)\}\,m + \{1+1\}\,n + \{1 - \surd(-1)\}\,r] \\ &= 2\,(m+r)\,\{(2n+m+r)^2 + (m-r)^2\} \\ &= 4\,(m+r)\,\{(m+n)^2 + (n+r)^2\}. \end{aligned}$$

That $m + r$ is a factor of the determinant in § 2 is seen by adding the first and last rows.

7. Another mode of verifying the resolvability of V is to increase the first row by all the others, arriving readily at

$$V(v, w, x, y) = \begin{vmatrix} v & w & x & y \\ -y & v+w+x+y & -v & -w \\ -x & -y & v+w+x+y & -v \\ -w & -x & -y & v+w+x+y \end{vmatrix};$$

then to perform the operation

$$(1-\epsilon)\,\mathrm{col}_1 + (1-\epsilon^2)\,\mathrm{col}_2 + (1-\epsilon^3)\,\mathrm{col}_3 + (1-\epsilon^4)\,\mathrm{col}_4,$$

when it is found that the first column has become such that the factor

$$(1-\epsilon)v + (1-\epsilon^2)w + (1-\epsilon^3)x + (1-\epsilon^4)y,$$

can be removed from it, and the elements

$$1, \ -\epsilon^2, \ -\epsilon^3, \ -\epsilon^4$$

left in the column.

8. From the irrational factors of V_n rational factors must arise by grouping, save when $n+1$ is prime. The formal result in regard to this is that *the rational factors of V_n are in number the same of those of $(x^{n+1}-1) \div (x-1)$, and are similar in degree.* Thus when n is 3, we have

$$\frac{x^4-1}{x-1} = (x+1)(x^2+1),$$

and hence the linear and quadratic factor of § 6. When n is 5,

$$\frac{x^6-1}{x-1} = (x+1)(x^2+x+1)(x^2-x+1)$$

and V_5 is resolvable into a linear factor and two quadratics. When n is 4 the V-function, $V(l, m, n, r)$ say, is irresolvable: and we have for its equivalent

$$\begin{aligned}
&l^4+m^4+n^4+r^4 \\
&+3(l^3m+m^3r+n^3l+r^3n)+2(l^3n+m^3l+n^3r+r^3m) \\
&\qquad\qquad\qquad\qquad\qquad\qquad +(l^3r+m^3n+n^3m+r^3l) \\
&+4(l^2+r^2)(m^2+n^2)+l^2r^2+m^2n^2 \\
&+7(l^2mn+lm^2r+ln^2r+mnr^2)+6(l^2mr+m^2nr+n^2lm+r^2nl) \\
&\qquad\qquad\qquad\qquad\qquad\qquad +4(l^2nr+m^2lr+n^2mr+r^2lm) \\
&+11lmnr.
\end{aligned}$$

9. In order to compare V with Boole's and other unisignants that have no negative terms in the elements we increase each row by the sum of all the rows in front of it, and thereafter perform on the columns the same operation. The result is

$$V(v, w, x, y) = \begin{vmatrix} v+w+x+y & w+x+y & x+y & y \\ v+w+x & v+2w+2x+y & w+2x+y & x+y \\ v+w & v+2w+x & v+2w+2x+y & w+x+y \\ v & v+w & v+w+x & v+w+x+y \end{vmatrix}.$$

Equivalents of the like kind for U and W can similarly be found.

10. It is also worth noting in conclusion that the converse of the property with which we started also holds, namely, *if the primary minors of a circulant be equal and non-zero, the sum of the elements must vanish.* For it is known that generally

$$C(a, b, c, d) = (a + b\omega + c\omega^2 + d\omega^3)(A + B\omega^3 + C\omega^2 + D\omega),$$

and therefore with the data just mentioned the second factor on the right would be $A(1 + w + w^2 + w^3)$, which is 0; so that the other general identity

$$C(a, b, c, d) = aA + bB + cC + dD$$

becomes
$$0 = (a + b + c + d) A,$$

whence the desired result.

Capetown, S.A.
7th Nov., 1915.

THE TWISTED CUBIC OF CONSTANT TORSION.

By *W. H. Salmon.*

THE following seems to be a shorter and more direct method than any yet published of arriving at the general equations of the cubic of constant torsion originally discovered by M. Lyon.* These equations, by an appropriate choice of origin, axes and parameter, are here obtained in their simplest form, consistent with perfect generality, and refer the curve, itself imaginary, to real axes.

The Cartesian coordinates x, y, z of the general unicursal twisted curve of the n^{th} degree can be written in the form

$$x = \frac{f_1(t)}{F(t)}, \quad y = \frac{f_2(t)}{F(t)}, \quad z = \frac{f_3(t)}{F(t)} \quad \ldots\ldots\ldots\ldots (1),$$

where F, f_1, f_2, f_3 are polynomials of degree n of a single parameter t. If the common denominator F be divided into the numerators f_1, f_2, f_3 giving a numerical quotient, the remainder is a polynomial of degree $n - 1$, and we can, by a change of origin, write the equations of a unicursal twisted curve of degree n in the form

$$x = \frac{u}{F}, \quad y = \frac{v}{F}; \quad z = \frac{w}{F} \quad \ldots\ldots\ldots\ldots\ldots (2),$$

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* *Annales de l'Enseignement Supérieur de Grenoble*, t. II., p. 353, 1890.

where u, v, w are polynomials in t of degree $n-1$, and F a polynomial of degree n.

The torsion τ at any point of the curve will be given by

$$\tau = \frac{\dddot{x}(\dot{y}\ddot{z} - \ddot{y}\dot{z}) + \dddot{y}(\dot{z}\ddot{x} - \ddot{z}\dot{x}) + \dddot{z}(\dot{x}\ddot{y} - \ddot{x}\dot{y})}{(\dot{y}\ddot{z} - \ddot{y}\dot{z})^2 + (\dot{z}\ddot{x} - \ddot{z}\dot{x})^2 + (\dot{x}\ddot{y} - \ddot{x}\dot{y})^2} \quad \ldots\ldots\ldots(3),$$

where $\dot{x}$, $\dot{y}$, $\dot{z}$, ..., denote $\frac{dx}{dt}, \frac{dy}{dt}, \frac{dz}{dt}, \ldots$.

Now $\dot{x} = \frac{\dot{u}F - \dot{F}u}{F^2}$, $\dot{y} = \frac{\dot{v}F - \dot{F}v}{F^2}$, $\dot{z} = \frac{\dot{w}F - \dot{F}w}{F^2}$,

$$\ddot{x} = \frac{\ddot{u}F^2 - 2\dot{u}F\dot{F} - uF\ddot{F} + 2u\dot{F}^2}{F^3},$$

with similar expressions for $\ddot{y}$ and $\ddot{z}$.

Therefore $\dot{y}\ddot{z} - \ddot{y}\dot{z} = 1/F^3\{(\dot{v}\ddot{w})F + (\ddot{v}w)\dot{F} + (v\dot{w})\ddot{F}\}$

where $(\dot{v}\ddot{w})$, ..., denotes $\dot{v}\ddot{w} - \ddot{v}\dot{w}$, ...,

and $\dddot{x} = 1/F^4\{\dddot{u}F^3 - 3\ddot{u}F^2\dot{F} - 3\dot{u}F^2\ddot{F} + 6\dot{u}F\dot{F}^2$
$$+ 6uF\dot{F}\ddot{F} - uF^2\dddot{F} - 6u\dot{F}^3\}.$$

Since $\Sigma\dddot{x}(\dot{y}\ddot{z} - \ddot{y}\dot{z}) = \tau\Sigma(\dot{y}\ddot{z} - \ddot{y}\dot{z})^2$,
therefore

$$\begin{aligned}\Sigma\{&\dddot{u}(\dot{v}\ddot{w})F^4 + \dddot{u}(v\dot{w})F^3\ddot{F} + \dddot{u}(\ddot{v}w)F^3\dot{F} - 3\ddot{u}(v\dot{w})F^2\dot{F}\ddot{F} \\ &- 3\dot{u}(\ddot{v}w)F^2\dot{F}\ddot{F} + 6\dot{u}(\ddot{v}w)F\dot{F}^3 + 6u(\dot{v}\ddot{w})F^2\dot{F}\ddot{F} \\ &- u(\dot{v}\ddot{w})F^3\dddot{F} - 6u(\dot{v}\ddot{w})F\dot{F}^3\} \\ &= \tau F\Sigma\{(\dot{v}\ddot{w})F + (\ddot{v}w)\dot{F} + (v\dot{w})\ddot{F}\}^2.\end{aligned}$$

But $\Sigma u(\dot{v}\ddot{w}) = \Sigma\dot{u}(\ddot{v}w) = \Sigma\ddot{u}(v\dot{w})$,

therefore $F^2\Sigma\{\dddot{u}[(\dot{v}\ddot{w})F + (\ddot{v}w)\dot{F} + (v\dot{w})\ddot{F}] - u(\dot{v}\ddot{w})\dddot{F}\}$
$$= \tau\Sigma\{(\dot{v}\ddot{w})F + (\ddot{v}w)\dot{F} + (v\dot{w})\ddot{F}\}^2 \ldots\ldots\ldots(4).$$

In the case of the cubic, equations (1) are

$$x = \frac{a_0t^2 + a_1t + a_2}{d_0t^3 + d_1t^2 + d_2t + d_3} = \frac{u}{F},$$

$$y = \frac{b_0t^2 + b_1t + b_2}{d_0t^3 + d_1t^2 + d_2t + d_3} = \frac{v}{F},$$

$$z = \frac{c_0t^2 + c_1t + c_2}{d_0t^3 + d_1t^2 + d_2t + d_3} = \frac{w}{F}.$$

Without loss of generality we may take $d_0 = 1$, and, increasing the parameter t by a constant, we may also make d_1 zero. Hence we may write $F = t^3 + dt + e$. In the above equations the origin is on the curve at the point where t is infinite; if in addition we take the axes along the tangent, the principal normal and the binormal at this point, then $b_0 = 0$, $c_0 = 0$, $c_1 = 0$. Therefore the equations of the general twisted cubic may be written

$$\left.\begin{aligned} x &= \frac{a_0 t^2 + a_1 t + a_2}{t^3 + dt + e} = \frac{u}{F}, \\ y &= \frac{b_1 t + b_2}{t^3 + dt + e} = \frac{v}{F}, \\ z &= \frac{c_2}{t^3 + dt + e} = \frac{w}{F}, \end{aligned}\right\} \quad \ldots\ldots\ldots\ldots (5).$$

Hence

$$\begin{array}{llll} u = a_0 t^2 + a_1 t + a_2, & \dot{u} = 2a_0 t + a_1, & \ddot{u} = 2a_0, & \dddot{u} = 0; \\ v = b_1 t + b_2, & \dot{v} = b_1, & \ddot{v} = 0, & \dddot{v} = 0; \\ w = c_2, & \dot{w} = 0, & \ddot{w} = 0, & \dddot{w} = 0; \\ F = t^3 + dt + e, & \dot{F} = 3t^2 + d, & \ddot{F} = 6t, & \dddot{F} = 6. \end{array}$$

Therefore

$$\begin{array}{lll} (\dot{v}\ddot{w}) = 0, & (\dot{w}\ddot{u}) = 0, & (\dot{u}\ddot{v}) = -2a_0 b_1; \\ (\ddot{v}w) = 0, & (\ddot{w}u) = -2a_0 c_2, & (\ddot{u}v) = 2a_0 (b_1 t + b_2); \\ (v\dot{w}) = -b_1 c_2, & (w\dot{u}) = c_2 (2a_0 t + a_1), & (u\dot{v}) = -a_0 b_1 t^2 - 2a_0 b_2 t + a_2 b_1 - a_1 b_2; \end{array}$$

and equation (4) becomes

$$\begin{aligned} 12a_0 b_1 c_2 F^3 = \tau\{& 4a_0^2 b_1^2 F^2 + (4a_0^2 b_1^2 t^2 + 8a_0^2 b_1 b_2 t + 4a_0^2 b_2^2 + 4a_0^2 c_2^2)\dot{F}^2 \\ & + (a_0^2 b_1^2 t^4 + 4a_0^2 b_1 b_2 t^3 + 4a_0^2 b_2^2 t^2 - 2a_0 a_2 b_1^2 t^2 + 2a_0 a_1 b_1 b_2 t^2 \\ & + 4a_0^2 c_2^2 t^2 - 4a_0 a_2 b_1 b_2 t + 4a_0 a_1 b_2^2 t + 4a_0 a_1 c_2^2 t + a_2^2 b_1^2 - 2a_1 a_2 b_1 b_3 \\ & + a_1^3 b_2^2 + a_1^2 c_2^2 + b_1^2 c_2^2)\ddot{F}^2 + (-8a_0^2 b_1^2 t - 8a_0^2 b_1 b_3) F\dot{F} \\ & + (4a_0^2 b_1^2 t^2 + 8a_0^2 b_1 b_2 t - 4a_0 a_2 b_1^2 + 4a_0 a_1 b_1 b_2) F\ddot{F} \\ & + (-4a_0^3 b_1^2 t^3 - 12a_0^2 b_1 b_2 t^2 - 8a_0^2 b_2^2 t + 4a_0 a_2 b_1^2 t - 4a_0 a_1 b_1 b_2 t \\ & - 8a_0^2 c_2^2 t + 4a_0 a_2 b_1 b_2 - 4a_0 a_1 b_2^2 - 4a_0 a_1 c_2^2)\dot{F}\ddot{F}\} \ldots\ldots\ldots\ldots (6). \end{aligned}$$

Equating the coefficients of t^5 on both sides we have $a_0^2 b_1 b_2 = 0$. But $a_0 = 0$ or $b_1 = 0$ makes the curve plane; therefore for finite constant torsion it is necessary that $b_2 = 0$. Equating the constant term on both sides with the condition $b_2 = 0$, we have $12a_0 b_1 c_2 e^2 = \tau\{4a_0^2 b_1^2 e^2 + 4a_0^2 c_2^2 d^2\}$; but from the coefficients of t^6 we see that $12a_0 b_1 c_2 = \tau . 4a_0^2 b_1^2$. Hence $a_0^2 c_2^2 d^2 = 0$. Rejecting $a_0 = 0$ and $c_2 = 0$, either of which makes the curve plane, we have $d = 0$.

Hence equation (6) reduces to

$$\begin{aligned} 3a_0 b_1 c_2 \{t^6 + 2et^3 + e^2\} &= \tau\{a_0^2 b_1^2 (t^6 + 2et^3 + e^2) \\ &+ (a_0^2 b_1^2 t^2 + a_0^2 c_2^2)\, 9t^4 + (a_0^2 b_1^2 t^4 - 2a_0 a_2 b_1^2 t^2 \\ &+ 4a_0^2 c_2^2 t^2 + 4a_0 a_1 c_2^2 t + a_2^2 b_1^2 + a_1^2 c_2^2 + b_1^2 c_2^2)\, 9t^2 \\ &- 2a_0^2 b_1^2 t\,(3t^5 + 3et^2) + (a_0^2 b_1^2 t^2 - a_0 a_2 b_1^2)(6t^4 + 6et) \\ &+ (-a_0^2 b_1^2 t^3 + a_0 a_2 b_1^2 t - 2a_0^2 c_2^2 t - a_0 a_1 c_2^2)\, 18t^3\} \quad \ldots\ldots(7). \end{aligned}$$

Equating the other coefficients we have the following relations:—

$$\tau\, a_0^2 b_1^2 = 3a_0 b_1 c_2 \quad \ldots\ldots\ldots\ldots\ldots\ldots(8),$$

$$9a_0^2 c_2^2 - 6a_0 a_2 b_1^2 = 0 \quad \ldots\ldots\ldots\ldots\ldots\ldots(9),$$

$$\tau\,(a_0^2 b_1^2 e + 9a_0 a_1 c_2^2) = 3a_0 b_1 c_2 e \quad \ldots\ldots\ldots\ldots(10),$$

$$a_2^2 b_1^2 + b_1^2 c_2^2 + c_2^2 a_1^2 = 0 \quad \ldots\ldots\ldots\ldots\ldots\ldots(11),$$

$$a_0 a_2 b_1^2 e = 0 \quad \ldots\ldots\ldots\ldots\ldots\ldots(12).$$

From equation (12), since a_0, a_2, b_1 must all be finite, $e = 0$. Therefore from equation (10), $a_0 a_1 c_2^2 = 0$, and so $a_1 = 0$. Hence $a_2^2 + c_2^2 = 0$, $3a_0 c_2 = 2a_2 b_1^2$, and $\tau = 3c_2/a_0 b_1$. Thus $c_2 = \pm i a_2$ and $b_1 = \pm i \sqrt{(\frac{3}{2} a_0 a_3)}$.

Writing A for a_0, B for a_2, and t for $1/t$, we arrive at the following result:—

All twisted cubics of constant torsion are imaginary, and their general equations can, by a proper choice of origin, axes, and the parameter t, be written

$$x = At + Bt^3, \quad y = \pm\sqrt{(-\tfrac{3}{2}AB)}\,t^2, \quad z = \pm\sqrt{(-B^2)}\,t^3,$$

where A, B are finite, independent arbitrary constants. The magnitude τ of this constant torsion is $\sqrt{(6B)}/A\sqrt{A}$.

ON BRIGGS'S PROCESS FOR THE REPEATED EXTRACTION OF SQUARE ROOTS.

By *J. W. L. Glaisher.*

§ 1. For the original calculation of the logarithms of certain prime numbers Briggs used a method which required the repeated extraction of the given number a great number of times. For example, by extracting the square root of 10 fifty-four times in succession he found the resulting root to be

1.00000 00000 00000 12781 91493 20032 35.

This repeated extraction of square roots was very laborious, and in chapter viii. of his *Arithmetica Logarithmica* (1624) Briggs gave a method of proceeding by means of differences from one root to the next. The object of this paper is to examine this method with reference to the principles on which it rests and the use which Briggs made of it, and also to consider other methods to which Briggs might have been led by it.

§ 2. Briggs seems to have observed that the decimal part of each successive square root was approximately equal to $\frac{1}{2}$ of the decimal part of its predecessor, and that if each decimal part were subtracted from $\frac{1}{2}$ of its predecessor the differences so formed were such that each was approximately equal to $\frac{1}{4}$ of its predecessor, and that if second differences were formed by subtracting each first difference from $\frac{1}{4}$ of its predecessor, these second differences were such that each was approximately equal to $\frac{1}{8}$ of its predecessor, and that if third differences were formed by subtracting each second difference from $\frac{1}{8}$ of its predecessor, these third differences were such that each was approximately $\frac{1}{16}$ of its predecessor, and so on.

Briggs denoted the decimal part of the square root with which he starts by A, and the first, second, third . . . differences by B, C, D, . . . and he applied his method to the calculation of successive square roots in the following manner. He first extracted the square root of the given number continually, in the ordinary manner, a certain number of times, and from the square roots thus obtained he calculated the first, second, third . . . differences B, C, D, . . . until, to the number of places included, one of these differences

became insensible. Suppose that the F-difference is the first to disappear. Then starting with $\frac{1}{32}$ of the E-difference as a new E-difference he derived from it new D-, C-, B-differences and a new A, which was the next square root*.

In the example which Briggs gives he starts with the number 1.0077696 and extracts its square root 9 times successively, the value of its 512th root being found to be

$$1.00001\ 51164\ 65999\ 05672\ 95048\ 8\dagger,$$

and by forming the first, second, third . . . differences from this and the preceding square roots (viz. the 256th, 128th, &c.), he obtains the following values of B, C, D, and E;

$$B,\ .00000\ 00001\ 14253\ 77215\ 03190\ 9,$$

$$C,\ .00000\ 00000\ 00001\ 72711\ 97889\ 3,$$

$$D,\ .00000\ 00000\ 00000\ 00004\ 56894\ 3,$$

$$E,\ .00000\ 00000\ 00000\ 00000\ 00020\ 7\ ;$$

F being insensible to this number of places. Then starting with $\frac{1}{32}E$, that is, with

$$.00000\ 00000\ 00000\ 00000\ 00000\ 65,$$

he derives from it the values of D, C, B, and A for the next root, the value of A being found to be

$$.00000\ 75582\ 04436\ 30121\ 42907\ 60,$$

which therefore is the decimal part of the next root.

Having thus explained his method by working out an example, Briggs concludes his chapter by giving expressions for B, C, D . . . in terms of A as far as the term in A^{10}.

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* I use Briggs's letters A, B, C, D . . . exactly as he did, but my first, second, . . . differences are his second, third . . . differences; for, the square root being $1+A$, Briggs calls A the first difference, viz., it is the difference between the square root and unity; and the difference between A and half the previous A he calls B, &c.: but A is merely the decimal part of the initial quantity, and it seems more natural not to include it among the differences, as it does not belong to the system formed by the others.

Briggs had no notation for distinguishing the successive A's, B's, . . . such as is now afforded by suffixes, nor had he a notation for powers which put in evidence the quantity raised to the power, *e.g.* A^4 was denoted by (4).

† Although Briggs used decimal fractions and had a special notation for them (by underlining them), still he practically treats his numbers as integers in the course of work, *e.g.* he writes this number

10000,15116,46599,90567,29504,88.

The commas are used, as now, to divide into convenient groups a long succession of figures, but he starts with the first figure, not with the first decimal.

§ 3. In order to investigate the principle of Briggs's process it is convenient to distinguish the successive square roots and differences by suffixes. Denoting the original quantity whose root is to be continually extracted by a, it is supposed that a consists of unity followed by a decimal, viz. $a = 1 + A$ where A is a decimal. I denote the decimal parts of the successive square roots by A_1, A_2, ... (in accordance with which A is equivalent to A_0) so that

$$a^{\frac{1}{2}} = (1 + A)^{\frac{1}{2}} = 1 + A_1,$$

$$a^{\frac{1}{4}} = (1 + A)^{\frac{1}{4}} = 1 + A_2,$$

$$a^{\frac{1}{8}} = (1 + A)^{\frac{1}{8}} = 1 + A_3, \text{ \&c.},$$

and in general the 2^n-th root of a, that is of $1 + A$, is equal to $1 + A_n$.

If for brevity we put $h = \dfrac{1}{2^n}$, then

$$a^h = (1 + A)^h = 1 + A_n,$$

$$a^{2h} = (1 + A)^{2h} = 1 + A_{n-1}, \quad a^{4h} = (1 + A)^{4h} = 1 + A_{n-2}, \text{ \&c.},$$

$$a^{\frac{1}{2}h} = (1 + A)^{\frac{1}{2}h} = 1 + A_{n+1}, \quad a^{\frac{1}{4}h} = (1 + A)^{\frac{1}{4}h} = 1 + A_{n+2}, \text{ \&c.},$$

Denoting the first, second, third, . . . differences corresponding to A_n by B_n, C_n, D_n, ..., it follows from their definitions that these quantities are given by the equations

$$B_n = \tfrac{1}{2} A_{n-1} - A_n,$$

$$C_n = \tfrac{1}{4} B_{n-1} - B_n,$$

$$D_n = \tfrac{1}{8} C_{n-1} - C_n,$$

$$E_n = \tfrac{1}{16} D_{n-1} - D_n, \text{ \&c.}$$

§ 4. Let the expansion of a^h, that is of $(1 + A)^h$, in ascending powers of h be $1 + V_1 h + V_2 h^2 + V_3 h^3 + \text{\&c.}$ Then

$$A_n = V_1 h + V_2 h^2 + V_3 h^3 + V_4 h^4 + \text{\&c.},$$

whence $$A_{n-1} = V_1 2h + V_2 2^2 h^2 + V_3 2^3 h^3 + V_4 2^4 h^4 + \text{\&c.},$$

and therefore

$$B_n = V_2 h^2 + (2^2 - 1) V_3 h^3 + (2^3 - 1) V_4 h^4 + (2^4 - 1) V_5 h^5 + \text{\&c.}$$

Putting $2h$ for h, dividing by 2^2, and subtracting, we find

$$C_n = (2^2 - 1) V_3 h^3 + (2^3 - 1)(2^2 - 1) V_4 h^4 + (2^4 - 1)(2^3 - 1) V_5 h^5 + \text{\&c.}$$

and, similarly,

$$D_n = (2^3-1)(2^2-1)V_4h^4 + (2^4-1)(2^3-1)(2^2-1)V_5h^5 + (2^5-1)(2^4-1)(2^3-1)V_6h^6 + \&c.,$$

$$E_n = (2^4-1)(2^3-1)(2^2-1)V_5h^5 + (2^5-1)(2^4-1)(2^3-1)(2^2-1)V_6h^6 + (2^6-1)\ldots(2^3-1)V_7h^7 + \&c.^*$$

&c. &c.

§ 5. Now suppose that to the number of places included F_n is insensible. This expresses that $\frac{1}{32}E_{n-1} = E_n$. Using (E_{n+1}), (D_{n+1}), . . . to denote the new E, D, . . . calculated by Briggs's method, he takes $(E_{n+1}) = \frac{1}{32}E_n$. Thus

$$(E_{n+1}) = (2^4-1)(2^3-1)(2^2-1)V_5\frac{h^5}{2^5} + (2^5-1)\ldots(2^2-1)V_6\frac{h^6}{2^5} + (2^6-1)\ldots(2^3-1)V_7\frac{h^7}{2^5} + \&c.$$

The quantity (D_{n+1}) is obtained from the formula

$$(D_{n+1}) = \tfrac{1}{16}D_n - (E_{n+1}),$$

and therefore

$$(D_{n+1}) = (2^3-1)(2^2-1)V_4\frac{h^4}{2^4} + (2^4-1)(2^3-1)(2^2-1)V_5\frac{h^5}{2^5}$$

$$-\{(2^5-1)(2^4-1)(2^3-1)(2^2-1) - 2(2^5-1)(2^4-1)(2^3-1)\}V_6\frac{h^6}{2^5}$$

$$-\{(2^6-1)(2^5-1)(2^4-1)(2^3-1) - 2(2^6-1)(2^5-1)(2^4-1)\}V_7\frac{h^7}{2^5}$$

$$+ \&c.$$

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* These series are convergent; for the s^{th} term in the series for the r^{th} difference is

$$(2^{r+s-1}-1)(2^{r+s-2}-1)\ldots(2^s-1)V_{r+s}h^{r+s}.$$

Now, as will be shown in §12, $V_{r+s}h^{r+s} = \dfrac{\{\log_e(1+A_n)\}^{r+s}}{(r+s)!}$, the numerator of which is approximately equal to A_n^{r+s}, and A_n is approximately equal to $\dfrac{A}{2^n}$. Thus $V_{r+1}h^{r+1}$ is approximately equal to $\dfrac{A^{r+s}}{(r+s)!\,2^{(r+s)n}}$. The ratio between this term and the previous term is therefore approximately

$$\frac{2^{r+s-1}-1}{2^{s-1}-1}\cdot\frac{A}{(r+s)\,2^n},$$

which nearly $= \dfrac{A}{(r+s)\,2^{n-r}}$. The largest value of r is n, so that this factor is always less than $\dfrac{A}{r+n}$.

Proceeding in this manner we find

$$(C_{n+1}) = (2^2-1)\,V_3\frac{h^3}{2^3} + (2^3-1)(2^2-1)\,V_4\frac{h^4}{2^4} + (2^4-1)(2^3-1)\,V_5\frac{h^5}{2^5}$$

$$+\{(2^5-1)(2^4-1)(2^3-1)(2^2-1) - 2(2^5-1)(2^4-1)(2^3-1)$$

$$+2^2(2^5-1)(2^4-1)\}\,V_6\frac{h^6}{2^5} + \&c.,$$

$$(B_{n+1}) = V_2\frac{h^2}{2^2} + (2^2-1)\,V_3\frac{h^3}{2^3} + (2^3-1)\,V_4\frac{h^4}{2^4} + (2^4-1)\,V_5\frac{h^5}{2^5}$$

$$-\{(2^5-1)(2^4-1)(2^3-1)(2^2-1) - 2(2^5-1)(2^4-1)(2^3-1)$$

$$+2^2(2^5-1)(2^4-1) - 2^3(2^5-1)\}\,V_6\frac{h^6}{2^5} - \&c.,$$

$$(A_{n+1}) = V_1\frac{h}{2} + V_2\frac{h^2}{2^2} + V_3\frac{h^3}{2^3} + V_4\frac{h^4}{2^4} + V_5\frac{h^5}{2^5}$$

$$+\{(2^5-1)(2^4-1)(2^3-1)(2^2-1) - 2(2^5-1)(2^4-1)(2^3-1)$$

$$+2^2(2^5-1)(2^4-1) - 2^3(2^5-1) + 2^4\}\,V_6\frac{h^6}{2^5} - \&c.,$$

the general term being

$$\{(2^{r-1}-1)(2^{r-2}-1)(2^{r-3}-1)(2^{r-4}-1) - 2(2^{r-1}-1)(2^{r-2}-1)(2^{r-3}-1)$$

$$+2^2(2^{r-1}-1)(2^{r-2}-1) - 2^3(2^{r-1}-1) + 2^4\}\,V_r\frac{h^r}{2^5}.$$

§ 6. Thus, up to and including terms of the order h^5, (A_{n+1}) differs from A_n only by the substitution of $\frac{1}{2}h$ for h and, to this degree of accuracy, it is equal to the value of A_{n+1}, the square root of A_n.

Now the fact that F_n disappears (to the number of places included) shows that $(2^5-1)(2^4-1)(2^3-1)(2^2-1)\,V_6h^6$ may be neglected. The numerator $\{(2^5-1)(2^4-1)(2^3-1)(2^2-1) - 2(2^5-1)(2^4-1)(2^3-1) + \&c.\}\,V_6h^6$ of the corresponding term in (A_{n+1}) is necessarily less than this quantity, and, in addition, this numerator is divided by 32. Thus (A_{n+1}) differs from A_{n+1} by a quantity which is less than $\frac{1}{32}$th of the quantity which is insensible, and therefore Briggs's rule gives the value of A_{n+1} with more than sufficient accuracy. The same is also true of B_{n+1}, C_{n+1}, &c.

§ 7. The quantities B_n, C_n, D_n, . . . may be expressed in terms of A_n and the previous A's, for

$$B_n = \tfrac{1}{2} A_{n-1} - A_n,$$

$$\begin{aligned} C_n &= \tfrac{1}{4} B_{n-1} - B_n, \\ &= \tfrac{1}{4}\left(\tfrac{1}{2} A_{n-2} - A_{n-1}\right) - \tfrac{1}{2} A_{n-1} + A_n, \\ &= \frac{1}{2^3} A_{n-3} - \frac{3}{2^2} A_{n-1} + A_n, \end{aligned}$$

and, similarly,

$$D_n = \frac{1}{2^6} A_{n-3} - \frac{7}{2^5} A_{n-2} + \frac{7}{2^3} A_{n-1} - A_n,$$

$$E_n = \frac{1}{2^{10}} A_{n-4} - \frac{15}{2^9} A_{n-3} + \frac{35}{2^7} A_{n-2} - \frac{15}{2^4} A_{n-1} + A_n,$$

$$F_n = \frac{1}{2^{15}} A_{n-5} - \frac{31}{2^{14}} A_{n-4} + \frac{155}{2^{12}} A_{n-3} - \frac{155}{2^9} A_{n-2} + \frac{31}{2^5} A_{n-1} - A_n,$$

$$G_n = \frac{1}{2^{21}} A_{n-6} - \frac{63}{2^{20}} A_{n-5} + \frac{651}{2^{18}} A_{n-4} - \frac{1395}{2^{15}} A_{n-3}$$

$$+ \frac{651}{2^{11}} A_{n-2} - \frac{63}{2^6} A_{n-1} + A_n.$$

In general, if Q_n be the r^{th} difference, then

$$Q_n = \frac{1}{2^{\frac{1}{2}r(r+1)}} \{(\eta - 2)(\eta - 2^2)(\eta - 2^3)\ldots(\eta - 2^r)\} A_n$$

where η is an operator such that $\eta^s A_n = A_{n-s}$.

§ 8. Briggs, however, does not express his B, C, D . . . in terms of A and the preceding A's, but in terms of A only. This, in the notation of the present paper, is equivalent to expressing B_n, C_n, D_n . . . in terms of A_n.

The values given by Briggs, which include terms up to A^{10}, are

$$B = \tfrac{1}{2} A^2,$$

$$C = \tfrac{1}{2} A^3 + \tfrac{1}{8} A^4,$$

$$D = \tfrac{7}{8} A^4 + \tfrac{7}{8} A^5 + \tfrac{7}{16} A^6 + \tfrac{1}{8} A^7 + \tfrac{1}{64} A^8,$$

$$E = 2\tfrac{5}{8} A^5 + 7 A^6 + 10\tfrac{15}{16} A^7 + 12\tfrac{69}{128} A^8 + 11\tfrac{11}{64} A^9 + 7\tfrac{105}{128} A^{10},$$

and so on, the values of I and K, the eighth and ninth differences, being

$$I = 54902\tfrac{89}{128}A^9 + 2558465\tfrac{77}{128}A^{10},$$

$$K = 2805527\tfrac{187}{256}A^{10}.*$$

§ 9. To obtain the values of $B, C, D, \ldots$ in this form we notice that $1 + A_{n-1} = (1 + A_n)^2$, $1 + A_{n-2} = (1 + A_n)^4$, &c., and therefore, from § 7,

$$B_n = \tfrac{1}{2}\{(1 + A_n)^2 - 1\} - A_n = \tfrac{1}{2}A_n^{\,2},$$

$$C_n = \frac{1}{2^3}\{(1 + A_n)^4 - 1\} - \frac{3}{2^2}\{(1 + A_n)^2 - 1\} + A_n,$$

$$D_n = \frac{1}{2^6}\{(1 + A_n)^8 - 1\} - \frac{7}{2^5}\{(1 + A_n)^4 - 1\} + \frac{7}{2^3}\{(1 + A_n)^2 - 1\} - A_n,$$

&c. &c.

§ 10. It is however more convenient to derive the expressions for C_n, $D_n \ldots$ in terms of A_n directly from their definitions $C_n = \frac{1}{4}B_{n-1} - B_n$, &c., in § 3 (*i.e.* to derive each difference from its predecessor) by making use of the fact that the change of the suffix n into $n - 1$ is equivalent to the change of A_n into $(1 + A_n)^2 - 1$, that is into $A_n(2 + A_n)$.

Thus,

$$C_n = \tfrac{1}{2}\{\tfrac{1}{4}A_n^{\,2}(2 + A_n)^2 - A_n^{\,2}\} = \tfrac{1}{2}A_n^{\,3} + \tfrac{1}{8}A_n^{\,4},$$

$$\begin{aligned} D_n &= \tfrac{1}{2}\{\tfrac{1}{8}A_n^{\,3}(2 + A_n)^3 - A_n^{\,3}\} + \tfrac{1}{8}\{\tfrac{1}{8}A^4(2 + A_n)^4 - A_n^{\,4}\} \\ &= \tfrac{7}{8}A_n^{\,4} + \tfrac{7}{8}A_n^{\,5} + \tfrac{7}{16}A_n^{\,6} + \tfrac{1}{8}A_n^{\,7} + \tfrac{1}{64}A_n^{\,8}. \end{aligned}$$

Similarly

$$E_n = \tfrac{7}{8}\{\tfrac{1}{16}A_n^{\,4}(2 + A_n)^4 - A_n^{\,4}\} + \tfrac{7}{8}\{\tfrac{1}{16}A_n^{\,5}(2 + A_n)^5 - A_n^{\,5}\} + \&c.,$$

or, as it may be conveniently written

$$\begin{aligned} E_n = \tfrac{7}{8}A_n^{\,4}\{(1 + \tfrac{1}{2}A_n)^4 - 1\} &+ \tfrac{7}{8}A_n^{\,5}\{2(1 + \tfrac{1}{2}A_n)^5 - 1\} \\ + \tfrac{7}{16}A_n^{\,6}\{2^2(1 + \tfrac{1}{2}A_n)^6 - 1\} &+ \tfrac{1}{8}A_n^{\,7}\{2^3(1 + \tfrac{1}{2}A_n)^7 - 1\} \\ &+ \tfrac{1}{64}A_n^{\,8}\{2^4(1 + \tfrac{1}{2}A_n)^8 - 1\}, \end{aligned}$$

which on reduction gives Briggs's value; and so on.

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* Briggs's values of the terms in A^{10} in I and K are inaccurate (in their fractional parts). These errors have been corrected in the values given above (See § 13).

These formulæ show that the complete expression for the r^{th} difference contains $2^r - r$ terms beginning with a term in A^{r-1}.

As has been mentioned the expressions given by Briggs extended only to the term in A^{10}.

§ 11. If we denote the $(r-1)^{\text{th}}$ difference by P_n, then

$$P_n = p_r A_n^{\;r} + p_{r+1} A_n^{\;r+1} + p_{r+2} A_n^{\;r+2} + \ldots,$$

and therefore the r^{th} difference Q_n is

$$Q_n = p_r A_n^{\;r} \{(1+\tfrac{1}{2}A_n)^r - 1\} + p_{r+1} A_n^{\;r+1} \{2(1+\tfrac{1}{2}A_n)^{r+1} - 1\}$$
$$+ p_{r+2} A_n^{\;r+2} \{2^2 (1+\tfrac{1}{2}A_n)^{r+2} - 1\} + \ldots.$$

If therefore we put

$$Q_n = q_{r+1} A_n^{\;r+1} + q_{r+2} A_n^{\;r+2} + q_{r+3} A_n^{\;r+3} + \ldots,$$

then

$$q_{r+1} = \frac{r}{2} p_r + p_{r+1},$$

$$q_{r+2} = \frac{(r)_2}{2^2} p_r + (r+1) p_{r+1} + 3 p_{r+2},$$

$$q_{r+3} = \frac{(r)_3}{2^3} p_r + \frac{(r+1)_2}{2} p_{r+1} + 2(r+2) p_{r+2} + 7 p_{r+3},$$

$$q_{r+4} = \frac{(r)_4}{2^4} p_r + \frac{(r+1)_3}{2^2} p_{r+1} + (r+2)_2 p_{r+2} + 2^2(r+3) p_{r+3} + 15 p_{r+4},$$

&c. &c.,

where $(r)_s$ denotes the coefficient of x^s in the expansion of $(1+x)^r$.

§ 12. By means of these formulæ the coefficients in any difference may be deduced from those of its predecessor; but the coefficients in any given difference P_n may be obtained directly as follows.

From § 4 we have

$$P_n = (2^{r-1}-1)(2^{r-2}-1)\ldots(2-1) V_r h^r + (2^r-1)\ldots(2^2-1) V_{r+1} h^{r+1} + \ldots$$

Now the V's are defined by the equation

$$(1+A)^h = 1 + V_1 h + V_2 h^2 + V_3 h^3 + \&\text{c.},$$

and the expansion of the quantity on the left-hand side is

$$1 + h \log_e(1+A) + \frac{\{h \log_e(1+A)\}^2}{2\,!} + \frac{\{h \log_e(1+A)\}^3}{3\,!} + \&c.,$$

so that
$$V_r = \frac{\{\log_e(1+A_n)\}^r}{r\,!}.$$

Thus

$$P_n = \frac{(2^{r-1}-1)(2^{r-2}-1)\dots(2-1)}{r\,!}\{\log_e(1+A_n)\}^r,$$

$$+\frac{(2^r-1)\ (2^{r-1}-1)\dots(2^2-1)}{(r+1)\,!}\{\log_e(1+A_n)\}^{r+1} + \&c.,$$

$$= \alpha_r\{\log_e(1+A_n)\}^r + \alpha_{r+1}\{\log_e(1+A_n)\}^{r+1} + \&c.,$$

where
$$\alpha_{r+m} = \frac{(2^{r+m-1}-1)\ (2^{r+m-2}-1)\dots(2^{m+1}-1)}{(r+m)\,!}.$$

Expanding the powers of $\log_e(1+A_n)$ we find that, if as before
$$P_n = p_r A_n^{\,r} + p_{r+1}A_n^{\,r+1} + p_{r+2}A_n^{\,r+2} + \&c.,$$
then

$$p_r\ = \alpha_r,$$

$$p_{r+1} = \alpha_{r+1} - \frac{r}{2}\alpha_r,$$

$$p_{r+2} = \alpha_{r+2} - \frac{r+1}{2}\alpha_{r+1} + \frac{r\,(3r+5)}{24}\alpha_r,$$

$$p_{r+3} = \alpha_{r+3} - \frac{r+2}{2}\alpha_{r+2} + \frac{(r+1)(3r+8)}{24}\alpha_{r+1} - \frac{r^3+5r^2+6r}{48}\alpha_r,$$

&c. &c.,

where

$$\alpha_r\ = \frac{(2^{r-1}-1)\ (2^{r-2}-1)\dots(2^2-1)}{r\,!},$$

$$\alpha_{r+1} = \frac{2^r-1}{r+1}\alpha_r,$$

$$\alpha_{r+2} = \frac{2^{r+1}-1}{(2^2-1)\ (r+2)}\alpha_{r+1},$$

$$\alpha_{r+3} = \frac{(2^{r+2}-1)}{(2^3-1)\ (r+3)}\alpha_{r+2},\ \&c.$$

§ 13. By means of the formulæ in the two preceding sections I have calculated the values of B_n, C_n, . . . in terms of A_n, thus verifying Briggs's values except in the case of five of the coefficients of A^{10} as mentioned below.

Suppressing the suffix throughout, the values of B_n, C_n . . . in terms of A_n are found to be

$$B = \tfrac{1}{2}A^2,$$

$$C = \tfrac{1}{2}A^3 + \tfrac{1}{8}A^4,$$

$$D = \tfrac{7}{8}A^4 + \tfrac{7}{8}A^5 + \tfrac{7}{16}A^6 + \tfrac{1}{8}A^7 + \tfrac{1}{64}A^8,$$

$$E = \tfrac{21}{8}A^5 + 7A^6 + \tfrac{175}{16}A^7 + \tfrac{1605}{128}A^8 + \tfrac{715}{64}A^9 + \tfrac{1001}{128}A^{10},$$

$$F = \tfrac{217}{16}A^6 + \tfrac{651}{8}A^7 + \tfrac{37975}{128}A^8 + \tfrac{106795}{128}A^9 + \tfrac{500123}{256}A^{10},$$

$$G = \tfrac{1953}{16}A^7 + \tfrac{193347}{128}A^8 + \tfrac{1468873}{128}A^9 + \tfrac{4375805}{64}A^{10},$$

$$H = \tfrac{248031}{128}A^8 + \tfrac{6035421}{128}A^9 + \tfrac{90476197}{128}A^{10},$$

$$I = \tfrac{7027545}{128}A^9 + \tfrac{327483597}{128}A^{10},$$

$$K = \tfrac{718215099}{256}A^{10}.\text{*}$$

Briggs expresses the coefficients as mixed fractions, and his coefficients of A^{10} in F, G, H, I, K are

$$1953\tfrac{285}{512},\ 68372\tfrac{79}{2048},\ 706845\tfrac{1493}{8192},\ 2558465\tfrac{23587}{32768},\ 2805527,$$

the true values in this form being

$$1953\tfrac{155}{256},\ 68371\tfrac{61}{64},\ 706845\tfrac{37}{128},\ 2558465\tfrac{77}{128},\ 2805527\tfrac{187}{256}.$$

§ 14. Briggs's formulæ enabled him to calculate B_n, C_n . . . from A_n alone, but it is not clear why he should have desired to do so. He mentions that B_n, C_n . . . can be so expressed,

---

* Although the coefficient of the leading term increases, the term itself decreases. For the first term of the r^{th} difference is $(2^r-1)(2^{r-1}-1)\ldots(2-1)V_{r+1}h^{r+1}$, and substituting approximate values, $V_{r+1}h^{r+1}$ approximately

$$= \frac{\{\log_e(1+A_n)\}^{r+1}}{(r+1)!} = \frac{A_n^{\,r+1}}{(r+1)!} = \frac{A^{r+1}}{(r+1)!\ 2^{(r+1)n}}.$$

Also $(2^r-1)(2^{r-1}-1)\ldots(2^2-1)$ is less than, and may for the present purpose be replaced by, $2^{\frac{1}{2}(r-1)(r+2)}$. Making these substitutions, the term becomes $\dfrac{1}{(r+1)!}\,\dfrac{A^{r+1}}{2^{(r+1)(n-\frac{1}{2}r)+1}}$, which diminishes with r. For the last (n^{th}) difference this value is

$$\frac{1}{(n+1)!}\,\frac{A^{n+1}}{2^{\frac{1}{2}n(n+1)+1}}.$$

and gives their values up to terms in A^{10}, without any indication of the manner in which they were obtained. He then applies the formulæ to calculate B_9, C_9, D_9, E_9 from A_9 to 30 places of decimals, A_9 being the decimal portion of the number 1.00001 51164 ... quoted in § 2.

It seems to me possible that Briggs having observed the curious result that B_n was equal to $\frac{1}{2}A_n^2$, was so led to express B_n, C_n, . . . in terms of A_n.

§ 15. The fundamental relations in § 3 show that

$$A_n = \tfrac{1}{2}A_{n-1} - \tfrac{1}{4}B_{n-1} + \tfrac{1}{8}C_{n-1} - \tfrac{1}{16}D_{n-1} + \ldots,$$

and therefore, if F_n may be neglected,

$$A_{n+1} = \tfrac{1}{2}A_n - \tfrac{1}{4}B_n + \tfrac{1}{8}C_n - \tfrac{1}{16}D_n + \tfrac{1}{32}E_n.$$

If therefore Briggs had only required A_{n+1}, he could have derived its value from those of B_n, C_n . . . without previously calculating B_{n+1}, C_{n+1} . . . , but these differences would have been required for the derivation of A_{n+2}.

§ 16. The values which Briggs obtained for B_n, C_n, . . . in terms of A_n would have enabled him to derive any A from its predecessor without calculating any differences. For, substituting in the last-written formula the values of B_n, C_n . . . in terms of A_n, we have

$$\begin{aligned} A_{n+1} &= \tfrac{1}{2}A_n - \tfrac{1}{4}\cdot\tfrac{1}{2}A_n^2 + \tfrac{1}{8}\left(\tfrac{1}{2}A_n^3 + \tfrac{1}{8}A_n^4\right) \\ &\quad - \tfrac{1}{16}\left\{\tfrac{7}{8}A_n^4 + \tfrac{7}{8}A_n^5 + \tfrac{7}{16}A_n^6 + \tfrac{1}{8}A_n^7 + \tfrac{1}{64}A_n^8\right\} + \tfrac{1}{32}\left\{\tfrac{21}{8}A_n^5 + \&c.\right\} \\ &= \tfrac{1}{2}A_n - \tfrac{1}{8}A_n^2 + \tfrac{1}{16}A_n^3 - \tfrac{5}{128}A_n^4 + \tfrac{7}{256}A_n^5, \end{aligned}$$

if terms beyond A^6 are neglected; and Briggs's values of the higher differences would have enabled him to extend this expression up to the term in A^{10}.

Briggs does not give this formula, and so presumably he did not obtain it. It may have escaped his notice, or it may be that as a calculator he preferred to work by differences, a method which he continually employed and of which he may almost be said to be the inventor.

Series were unknown in Briggs's time, but if he had noticed that A could be expressed in terms of the preceding A, B, C ... it would seem that he might have given the formula for A_{n+1} in terms of A_n in a finite form, rejecting powers of A which were insensible to the number of places included. There is however the important difference between

the expressions for B_n, C_n, ... in terms of A_n (which he does give) and of A_{n+1} in terms of A_n, that the complete expressions for the former are finite while the latter cannot be expressed in finite terms.

§ 17. The preceding expansion of A_{n+1} in terms of A_n of course follows at once from the formula

$$1 + A_{n+1} = (1 + A_n)^{\frac{1}{2}},$$

and similarly we have

$$\begin{aligned} A_{n+2} &= (1 + A_n)^{\frac{1}{4}} - 1 \\ &= \tfrac{1}{4}A_n - \tfrac{3}{32}A_n^2 + \tfrac{7}{128}A_n^3 - \&c. \end{aligned}$$

§ 18. As shown in § 12, the quantities V_1, V_2, V_3, ... are $\log_e(1+A)$, $\dfrac{\{\log_e(1+A)\}^2}{2\,!}$, $\dfrac{\{\log_e(1+A)\}^3}{3\,!}$, ..., that is, they are $\dfrac{(\log_e a)^2}{2!}$, $\dfrac{(\log_e a)^3}{3!}$, ...

Since $h = 2^{-n}$, it follows from the formulæ of § 4 that the limit, when n is very large, of $2^n A_n$ is $\log_e a$, and that the limits of $2^{2n}B_n$, $2^{3n}C_n$, $2^{4n}D_n$, ... are

$$\frac{(\log_e a)^2}{2!},\quad \frac{(2^2-1)}{3\,!}(\log_e a)^3,\quad \frac{(2^3-1)(2^2-1)}{4\,!}(\log_e a)^4,\ \ldots.$$

The first of these results, viz. that the limit of $2^n A_n$ is $\log_e a$, is involved in the formula by means of which Briggs obtained the logarithms of the early primes, and for which he calculated the value of A_n; for his actual process was equivalent to $\log_{10} a = 2^n A_n \times .43429...$, this multiplier .43429... being derived from the repeated extraction of the square root of 10, the logarithm of which was known.

§ 19. The differences B_n, C_n, . . . do not seem at the present day to possess mathematical interest of their own. They are derived from a system of successive square roots which were constructed in order to obtain as the final result a very high root, but which do not form a mathematical table that would be calculated for its own sake. I am afraid that the interest in the differences is almost entirely historic, and consists in the fact that their existence was discovered by Briggs, that he used them for calculating square roots, and that he obtained formulæ for them in terms of the decimal part of the final square root to which they were attached.

With reference to these formulæ it is curious that any mathematical work of so fine a calculator as Briggs should not have been quite free from error; but it is likely that he never used these formulæ as far as the term in A^{10}, and that when he wrote the Introduction to the *Arithmetica Logarithmica*, he had partly forgotten the details of his work connected with the calculation of these early primes, and printed the formulæ as he found them among his papers.

I may mention that Hutton has given a very good account of chapter viii. of the *Arithmetica Logarithmica*, which forms the subject of the present paper, in the Introduction to the numerous editions of his *Logarithms*. Also Delambre in vol. i. of his *Histoire de l'Astronomie Moderne* (pp. 538–541) has given a full account of Briggs's process and has quoted his formula for B, C, D, E in terms of A (§ 13). He makes no examination of Briggs's results and process, but after quoting the formulæ he remarks: "Le procédé précédent [*i.e.* the process] paraît bien préférable à ces formules. Briggs ne démontre rien, il paraît avoir trouvé le tout par le fait et d'après ses calculs; cependant, pour donner ces formules si longues, il a dû se faire une espèce de théorie empirique, dont il ne parle pas."

§ 20. If Briggs had repeatedly extracted cube roots instead of the square roots he would have found that similar differences existed and were capable of expression in a similar manner. For, proceeding as in §§ 3 and 9, if α be a decimal, and if

$$(1+\alpha)^{\frac{1}{3}} = 1+\alpha_1,\quad (1+\alpha)^{\frac{1}{9}} = 1+\alpha_2,\quad (1+\alpha)^{\frac{1}{27}} = 1+\alpha_3, \text{ \&c.},$$

and if

$$\begin{aligned} \beta_n &= \tfrac{1}{3}\alpha_{n-1} - \alpha_n, \\ \gamma_n &= \tfrac{1}{9}\beta_{n-1} - \beta_n, \\ \delta_n &= \tfrac{1}{27}\gamma_{n-1} - \gamma_n, \text{ \&c.,} \end{aligned}$$

then we find

$$\begin{aligned} \beta_n &= \alpha_n^2 + \tfrac{1}{3}\alpha_n^3, \\ \gamma_n &= \tfrac{8}{3}\alpha_n^3 + \tfrac{14}{3}\alpha_n^4 + \tfrac{14}{3}\alpha_n^5 + \tfrac{28}{9}\alpha_n^6 + \tfrac{4}{3}\alpha_n^7 + \tfrac{1}{3}\alpha_n^8 + \tfrac{1}{27}\alpha_n^9, \\ &\text{\&c.} \qquad\qquad \text{\&c.} \end{aligned}$$

DETERMINANTS OF CYCLICALLY REPEATED ARRAYS.

By *Sir Thomas Muir, LL.D.*

1. WRITERS like Puchta, Noether, W. Burnside,* who have dealt with determinants of the type here specified, have restricted themselves to cases where the circulating arrays were also themselves circulant, and where as a consequence the determinant is expressible as a product of linear factors. It will be found interesting to withdraw for a moment this restriction, and to see how it conduces to the discovery of additional properties of even the less general functions.

2. *The circulant of two* n-*line arrays is expressible as a product of two* n-*line determinants.* For, taking n equal to 3 and the arrays of $|a_1 b_2 c_3|$, $|a_4 b_5 c_6|$ as the two circulating arrays, the determinant in question is

$$\begin{vmatrix} a_1 & a_2 & a_3 & a_4 & a_5 & a_6 \\ b_1 & b_2 & b_3 & b_4 & b_5 & b_6 \\ c_1 & c_2 & c_3 & c_4 & c_5 & c_6 \\ a_4 & a_5 & a_6 & a_1 & a_2 & a_3 \\ b_4 & b_5 & b_6 & b_1 & b_2 & b_3 \\ c_4 & c_5 & c_6 & c_1 & c_2 & c_3 \end{vmatrix};$$

and this, when we reverse the order of the last three columns and thereafter the order of the last three rows, becomes

$$\begin{vmatrix} a_1 & a_2 & a_3 & a_6 & a_5 & a_4 \\ b_1 & b_2 & b_3 & b_6 & b_5 & b_4 \\ c_1 & c_2 & c_3 & c_6 & c_5 & c_4 \\ c_4 & c_5 & c_6 & c_3 & c_2 & c_1 \\ b_4 & b_5 & b_6 & b_3 & b_2 & b_1 \\ a_4 & a_5 & a_6 & a_3 & a_2 & a_1 \end{vmatrix},$$

which being centrosymmetric is resolvable into

$$\begin{vmatrix} a_1+a_4 & a_2+a_5 & a_3+a_6 \\ b_1+b_4 & b_2+b_5 & b_3+b_6 \\ c_1+c_4 & c_2+c_6 & c_3+c_6 \end{vmatrix} \cdot \begin{vmatrix} a_1-a_4 & a_2-a_5 & a_3-a_6 \\ b_1-b_4 & b_2-b_5 & b_3-b_6 \\ c_1-c_4 & c_2-c_5 & c_3-c_6 \end{vmatrix}.$$

* *Denkschr. . . . Akad. d. Wiss.* (Wien), vol. xxxviii., pp. 215–221 : vol. xliv., pp. 277–282; *Math. Annalen*, vol. xvi, pp. 322–325, 551–555; *Messenger of Math.*, vol. xxiii, pp. 112–114.

3. Taking the special case of the foregoing, where the circulating arrays are the arrays of the circulants $C(a_1, a_2, a_3)$, $C(a_4, a_5, a_6)$, we find the result of the resolution to be

$$C(a_1+a_4,\ a_2+a_5,\ a_3+a_6)\,.\,C(a_1-a_4,\ a_2-a_5,\ a_3-a_6),$$

whence there come six linear factors, in agreement with the result obtained by the writers above mentioned.

4. We may note in passing that this circulant of two three-line circulant arrays is not altered in substance by changing

$$a_1,\ a_2,\ a_3,\quad a_4,\quad a_5,\quad a_6$$

into

$$a_1,\ a_3,\ a_2,\ -a_4,\ -a_6,\ -a_5,$$

as is readily seen on changing rows into columns and multiplying by $(-1)^3\,.\,(-1)^3$. This being the case, if we take the product of the two forms, it must be possible to extract the square root of both sides; and, doing so, we find that *the circulant of two three-line circulant arrays is itself expressible as an ordinary three-line circulant*, namely,

$$C(U,\ V,\ W),$$

where

$$U \equiv a_1^2 + 2a_2a_3 - a_4^2 - 2a_5a_6,\qquad V \equiv a_3^2 + 2a_1a_2 - a_6^2 - 2a_4a_5,$$
$$W \equiv a_2^2 + 2a_1a_3 - a_5^2 - 2a_4a_6.$$

5. *The circulant of* n *two-line general arrays*

$$\begin{array}{cccc} a_1\ a_2 & a_3\ a_4 & a_5\ a_6 & \\ b_1\ b_2, & b_3\ b_4, & b_5\ b_6, & \ldots, \end{array}$$

is divisible by

$$\left|\begin{array}{cc} a_1+a_3+a_5+\ldots & a_2+a_4+a_6+\ldots \\ b_1+b_3+b_5+\ldots & b_2+b_4+b_6+\ldots \end{array}\right|.$$

If, merely for shortness in writing, we take n equal to 3, the circulant in question is

$$\left|\begin{array}{cccccc} a_1 & a_2 & a_3 & a_4 & a_5 & a_6 \\ b_1 & b_2 & b_3 & b_4 & b_5 & b_6 \\ a_5 & a_6 & a_1 & a_2 & a_3 & a_4 \\ b_5 & b_6 & b_1 & b_2 & b_3 & b_4 \\ a_3 & a_4 & a_5 & a_6 & a_1 & a_2 \\ b_3 & b_4 & b_5 & b_6 & b_1 & b_2 \end{array}\right|,$$

and this is seen to be equal to

$$\begin{vmatrix} a_1+a_3+a_5 & a_2+a_4+a_6 & a_3 & a_4 & a_5 & a_6 \\ b_1+b_3+b_5 & b_2+b_4+b_6 & b_3 & b_4 & b_5 & b_6 \\ a_5+a_1+a_3 & a_6+a_2+a_4 & a_1 & a_2 & a_3 & a_4 \\ b_5+b_1+b_3 & b_6+b_2+b_4 & b_1 & b_2 & b_3 & b_4 \\ a_3+a_5+a_1 & a_4+a_6+a_2 & a_5 & a_6 & a_1 & a_2 \\ b_3+b_5+b_1 & b_4+b_6+b_2 & b_5 & b_6 & b_1 & b_2 \end{vmatrix},$$

which on performing the operations

$$\mathrm{row}_6-\mathrm{row}_4, \quad \mathrm{row}_5-\mathrm{row}_3, \quad \mathrm{row}_4-\mathrm{row}_2, \quad \mathrm{row}_3-\mathrm{row}_1,$$

becomes resolvable into

$$\begin{vmatrix} a_1+a_3+a_5 & a_2+a_4+a_6 \\ b_1+b_3+b_5 & b_2+b_4+b_6 \end{vmatrix} \cdot \begin{vmatrix} a_1-a_3 & a_2-a_4 & a_3-a_5 & a_4-a_6 \\ b_1-b_3 & b_2-b_4 & b_3-b_5 & b_4-b \\ a_5-a_1 & a_6-a_2 & a_1-a_3 & a_2-a_4 \\ b_5-b_1 & b_6-b_2 & b_1-b_3 & b_2-b_4 \end{vmatrix}.$$

6. Taking the special case of the foregoing where the given two-line arrays are themselves circulant, that is to say, where

$$\begin{aligned} & b_1,\ b_2,\ b_3,\ b_4,\ b_5,\ b_6, \\ = {} & a_2,\ a_1,\ a_4,\ a_3,\ a_6,\ a_5, \end{aligned}$$

we see that the first factor in the result is

$$C(a_1+a_3+a_5,\ a_2+a_4+a_6),$$

and the second is the axisymmetric determinant

$$\begin{vmatrix} a_1-a_3 & a_2-a_4 & a_3-a_5 & a_4-a_6 \\ a_2-a_4 & a_1-a_3 & a_4-a_6 & a_3-a_5 \\ a_5-a_1 & a_6-a_2 & a_1-a_3 & a_2-a_4 \\ a_6-a_2 & a_5-a_1 & a_2-a_4 & a_1-a_3 \end{vmatrix}.$$

7. *The circulant of* n *two-line general arrays is equal to the determinant of four* n*-line circulant arrays.* This follows at once from advancing the odd-numbered columns to occupy the first n places, and thereafter treating the rows in the same manner.

For example, the circulant of the four arrays

$$\begin{matrix} a_1\, a_2 & a_3\, a_4 & a_5\, a_6 & a_7\, a_8 \\ b_1\, b_2, & b_3\, b_4, & b_5\, b_6, & b_7\, b_8, \end{matrix}$$

is equal to the determinant of the arrays of the four circulants

$$\begin{matrix} C(a_1, a_3, a_5, a_7), & C(a_2, a_4, a_6, a_8), \\ C(b_1, b_3, b_5, b_7), & C(b_2, b_4, b_6, b_8). \end{matrix}$$

8. If the four given two-line arrays in the preceding paragraph be made circulants the arrays of $C(b_1, b_3, b_5, b_7)$, $C(b_2, b_4, b_6, b_8)$ become the arrays of

$$C(a_2, a_4, a_6, a_8), \quad C(a_1, a_3, a_5, a_7),$$

and the resulting determinant becomes the circulant of two four-line circulant arrays, and as such resolvable into linear factors. Note must be taken, however, that it is not the eight-line determinant considered by Puchta in his first memoir. Both are eight-line determinants which are circulants of circulant arrays; but, while the one here appearing is the circulant of four two-line circulant arrays, Puchta's is the circulant of two four-line arrays each of which is a circulant of two two-line circulant arrays. The distinction between them is perhaps more simply brought out by viewing them as eliminants, the one being the eliminant of

$$\left.\begin{matrix} (1, x, y, xy, y^2, xy^2, y^3, xy^3 \between a_1, a_2, a_3, \ldots, a_8) = 0 \\ x^2 = 1, \ y^4 = 1 \end{matrix}\right\},$$

and the other, Puchta's, the eliminant of

$$\left.\begin{matrix} (1, x, y, xy, z, xz, yz, xyz \between a_1, a_2, a_3, \ldots, a_8) = 0 \\ x^2 = 1, \ y^2 = 1, \ z^2 = 1 \end{matrix}\right\}.$$

Further, if in writing the eight-termed equations here, we use

$$(1, x)\,(1, y, y^2, y^3)$$

to stand for the first set of facients, so that

$$(1, x)\,(1, y)\,(1, z)$$

stands for the second set, we have an additional aid to clearness.

9. *The circulant of* n *three-line general arrays*

$$\begin{matrix} a_1\, a_2\, a_3 & a_4\, a_5\, a_6 & a_7\, a_8\, a_9 & \\ b_1\, b_2\, b_3 & b_4\, b_5\, b_6 & b_7\, b_8\, b_9 & \\ c_1\, c_2\, c_3, & c_4\, c_5\, c_6, & c_7\, c_8\, c_9, & \ldots, \end{matrix}$$

is divisible by

$$\begin{vmatrix} a_1+a_4+a_7+\dots & a_2+a_5+a_8+\dots & a_3+a_6+a_9+\dots \\ b_1+b_4+b_7+\dots & b_2+b_5+b_8+\dots & b_3+b_6+b_9+\dots \\ c_1+c_4+c_7+\dots & c_2+c_5+c_8+\dots & c_3+c_6+c_9+\dots \end{vmatrix}.$$

The proof of this is quite similar to that in §5, and the two together at once suggest the formulating of a wide generalization.

When n is 3 the cofactor is

$$\begin{vmatrix} a_1-a_4 & a_2-a_5 & a_3-a_6 & a_4-a_7 & a_5-a_8 & a_6-a_9 \\ b_1-b_4 & b_2-b_5 & b_3-b_6 & b_4-b_7 & b_5-b_8 & b_6-b_9 \\ c_1-c_4 & c_2-c_5 & c_3-c_6 & c_4-c_7 & c_5-c_8 & c_6-c_9 \\ a_7-a_1 & a_8-a_2 & a_9-a_3 & a_1-a_4 & a_2-a_5 & a_3-a_6 \\ b_7-b_1 & b_8-b_2 & b_9-b_3 & b_1-b_4 & b_2-b_5 & b_3-b_6 \\ c_7-c_1 & c_8-c_2 & c_9-c_3 & c_1-c_4 & c_2-c_5 & c_3-c_6 \end{vmatrix},$$

which like that at the close of §5 is conveniently viewable as the determinant of four arrays, two of which are identical.

10. *The circulant of* n *three-line general arrays is equal to the determinant of nine (i.e.* 3^2*)* n*-line circulant arrays.* For example, when n is 3, the circulant of the arrays of

$$|a_1b_2c_3|, \quad |a_4b_5c_6|, \quad |a_7b_8c_9|,$$

is equal to the determinant of the arrays of the nine circulants

$$\begin{array}{lll} C(a_1, a_4, a_7), & C(a_2, a_5, a_8), & C(a_3, a_6, a_9), \\ C(b_1, b_4, b_7), & C(b_2, b_5, b_8), & C(b_3, b_6, b_9), \\ C(c_1, c_4, c_7), & C(c_2, c_5, c_8), & C(c_3, c_6, c_9). \end{array}$$

This is established exactly in the manner of §7, and the general proposition which includes the two is readily grasped.

11. By making the given arrays in §§ 9, 10 circulant arrays we learn that *the circulant of the arrays of*

$$C(a_1, a_2, a_3), \quad C(a_4, a_5, a_6), \quad C(a_7, a_8, a_9),$$

is divisble by

$$C(a_1+a_4+a_7, \ a_2+a_5+a_8, \ a_3+a_6+a_9), \qquad (\alpha),$$

and that *the circulant of the arrays of*

$$C(a_1, a_2, a_3), \quad C(a_4, a_5, a_6), \quad C(a_7, a_8, a_9),$$

is equal to the circulant of the arrays of

$$C(a_1, a_4, a_7),\quad C(a_2, a_5, a_8),\quad C)a_3, a_6, a_9), \qquad (\beta).$$

From the latter result it follows that *the said circulant*, $C_{3.3}$ say, *is invariant to the interchange of*

$$\left.\begin{array}{l} a_2 \textit{ with } a_4 \\ a_3 \textit{ with } a_7 \\ a_6 \textit{ with } a_8 \end{array}\right\} \qquad (\gamma),$$

and from this and the former result that $C_{3.3}$ *is divisible by*

$$C(a_1 + a_2 + a_3,\ a_4 + a_5 + a_6,\ a_7 + a_8 + a_9), \qquad (\delta).$$

Naturally each of these four results can be established independently.

12. It is important, however, now to note that the theorem of § 9 is susceptible of extension in a quite different direction when the number of given arrays is the same as the number of elements in each array. For the case where this common number is 3 the wider theorem is:—*the circulant of the three arrays*

$$\begin{array}{ccc} a_1\ a_2\ a_3 & a_4\ a_5\ a_6 & a_7\ a_8\ a_9 \\ b_1\ b_2\ b_3 & b_4\ b_5\ b_6 & b_7\ b_8\ b_9 \\ c_1\ c_2\ c_3, & c_4\ c_5\ c_6, & c_7\ c_8\ c_9, \end{array}$$

is divisible by

$$\begin{vmatrix} a_1 + a_4\gamma + a_7\gamma^2 & a_2 + a_5\gamma + a_8\gamma^2 & a_3 + a_6\gamma + a_9\gamma^2 \\ b_1 + b_4\gamma + b_7\gamma^2 & b_2 + b_5\gamma + b_8\gamma^2 & b_3 + b_6\gamma + b_9\gamma^2 \\ c_1 + c_4\gamma + c_7\gamma^2 & c_2 + c_5\gamma + c_8\gamma^2 & c_3 + c_6\gamma + c_9\gamma^2 \end{vmatrix},$$

where γ *is any third root of* 1. On the given determinant

$$\begin{vmatrix} a_1 & a_2 & a_3 & a_4 & a_5 & a_6 & a_7 & a_8 & a_9 \\ b_1 & b_2 & b_3 & b_4 & b_5 & b_6 & b_7 & b_8 & b_9 \\ c_1 & c_2 & c_3 & c_4 & c_5 & c_6 & c_7 & c_8 & c_9 \\ a_7 & a_8 & a_9 & a_1 & a_2 & a_3 & a_4 & a_5 & a_6 \\ b_7 & b_8 & b_9 & b_1 & b_2 & b_3 & b_4 & b_5 & b_6 \\ c_7 & c_8 & c_9 & c_1 & c_2 & c_3 & c_4 & c_5 & c_6 \\ a_4 & a_5 & a_6 & a_7 . a_8 & a_9 & a_1 & a_2 & a_3 \\ b_4 & b_5 & b_6 & b_7 & b_8 & b_9 & b_1 & b_2 & b_3 \\ c_4 & c_5 & c_6 & c_7 & c_8 & c_9 & c_1 & c_2 & c_3 \end{vmatrix}$$

we first perform the operations

$$\mathrm{col}_1 + \gamma\,\mathrm{col}_4 + \gamma^2\,\mathrm{col}_7,\quad \mathrm{col}_2 + \gamma\,\mathrm{col}_5 + \gamma^2\,\mathrm{col}_8,\quad \mathrm{col}_3 + \gamma\,\mathrm{col}_6 + \gamma^2\,\mathrm{col}_9,$$

and then on the resulting determinant the operations

$$\mathrm{row}_9 - \gamma\,\mathrm{row}_6,\quad \mathrm{row}_8 - \gamma\,\mathrm{row}_5,\quad \mathrm{row}_7 - \gamma\,\mathrm{row}_4,$$

followed by

$$\mathrm{row}_6 - \gamma\,\mathrm{row}_3,\quad \mathrm{row}_5 - \gamma\,\mathrm{row}_2,\quad \mathrm{row}_4 - \gamma\,\mathrm{row}_1.$$

The result of this is that 0 appears in every place of the first three columns except those situated in the first three rows; and the determinant of the non-zero elements being

$$|a_1 + a_4\gamma + a_7\gamma^2 \quad b_2 + b_5\gamma + b_8\gamma^2 \quad c_3 + c_6\gamma + c_9\gamma^2|,$$

is, as expected, a factor of the original determinant. The cofactor is

$$\left|\begin{matrix} a_1 - \gamma a_4 & a_2 - \gamma a_5 & a_3 - \gamma a_6 & a_4 - \gamma a_7 & a_5 - \gamma a_8 & a_6 - \gamma a_9 \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \end{matrix}\right|,$$

differing from that of § 9 merely in having γ prefixed to the second term of every element.

13. As we shall presently see that the three determinants, got from the three-line factor by giving γ its three values, do not have a common factor, there follows the important theorem that *the circulant of the three arrays of* $|a_1 b_2 c_3|$, $|a_4 b_5 c_6|$, $|a_7 b_8 c_9|$ *is equal to the product of the three determinants*

$$|a_1 + a_4 + a_7 \quad b_2 + b_5 + b_8 \quad c_3 + c_6 + c_9|,$$

$$|a_1 + a_4\gamma + a_7\gamma^2 \quad b_2 + b_5\gamma + b_8\gamma^2 \quad c_3 + c_6\gamma + c_9\gamma^2|,$$

$$|a_1 + a_4\gamma^2 + a_7\gamma \quad b_2 + b_5\gamma^2 + b_8\gamma \quad c_3 + c_6\gamma^2 + c_9\gamma|,$$

where γ *is a prime third root* of 1.

14. Taking the second of these determinant factors, namely,

$$\left|\begin{matrix} a_1 + a_4\gamma + a_7\gamma^2 & a_2 + a_5\gamma + a_8\gamma^2 & a_3 + a_6\gamma + a_9\gamma^2 \\ b_1 + b_4\gamma + b_7\gamma^2 & b_2 + b_5\gamma + b_8\gamma^2 & b_3 + b_6\gamma + b_9\gamma^2 \\ c_1 + c_4\gamma + c_7\gamma^2 & c_2 + c_5\gamma + c_8\gamma^2 & c_3 + c_6\gamma + c_9\gamma^2 \end{matrix}\right|,$$

and seeking to express it as a sum of determinants with monomial elements we find that there are nine of the determinants free of γ, nine with γ as a factor, and nine with γ^2 as

a factor. If we denote each of the twenty-seven by the suffixes occurring in it, for example,

$$|a_4 b_5 c_9| \text{ by } 459,$$

the result of the development is

$$\begin{aligned}&(123+456+789+159+267+348-168-249-357)\\ &+(126-135+234+189-279+378+459-468+567)\gamma\\ &+(129-138+237+156-246+345+489-579+678)\gamma^2:\end{aligned}$$

or, say $$P+Q\gamma+R\gamma^2.$$

With this notation it follows that the product at the end of §13 must be

$$(P+Q+R)(P+Q\gamma+R\gamma^2)(P+Q\gamma^2+R\gamma),$$

and this we know to be equal to the circulant

$$\begin{vmatrix} P & Q & R \\ R & P & Q \\ Q & R & P \end{vmatrix}.$$

We thus have the theorem that *the circulant of the arrays of*

$$|a_1 b_2 c_3|,\quad |a_4 b_5 c_6|,\quad |a_7 b_8 c_9|,$$

is expressible as an ordinary three-line circulant

$$C(P,\ Q,\ R),$$

where P, Q, R are aggregates of nine determinants whose columns are taken from the arrays

$$\begin{matrix} a_1\, a_4\, a_7 & a_2\, a_5\, a_8 & a_3\, a_6\, a_9 \\ b_1\, b_4\, b_7 & b_2\, b_5\, b_8 & b_3\, b_6\, b_9 \\ c_1\, c_4\, c_7, & c_2\, c_5\, c_8, & c_3\, c_6\, c_9, \end{matrix}$$

one from each.

15. Another notation for the determinants in P, Q, R would be that in which lmn would denote the determinant whose columns are the l^{th} of $|a_1 b_4 c_7|$, the m^{th} of $|a_2 b_5 c_8|$ and the n^{th} of $|a_3 b_6 c_9|$. We should then have

$$\begin{aligned}P&=111+123+132+213+222+231+312+321+333,\\ Q&=112+121+133+211+223+232+313+322+331,\\ R&=113+122+131+212+221+233+311+323+332,\end{aligned}$$

where the sum of the integers specifying a determinant is in P of the form $3r$, in Q of the form $3r+1$, and in R of the form $3r+2$.

16. Continuing now the specialization interrupted at the end of § 11 we learn from § 13 that *the circulant $C_{3.3}$, of the arrays of $C(a_1, a_2, a_3)$, $C(a_4, a_5, a_6)$ $C(a_7, a_8, a_9)$ is equal to the product of three circulants*

$$\begin{aligned}&C(a_1+a_4+a_7, \quad a_2+a_5+a_8, \quad a_3+a_6+a_9)\\ \cdot &C(a_1+a_4\gamma+a_7\gamma^2, \; a_2+a_5\gamma+a_8\gamma^2, \; a_3+a_6\gamma+a_9\gamma^2)\\ \cdot &C(a_1+a_4\gamma^2+a_7\gamma, \; a_2+a_5\gamma^2+a_8\gamma, \; a_3+a_6\gamma^2+a_9\gamma),\end{aligned}$$

and therefore, by the allowable interchange, also equal to the product of other three circulants

$$\begin{aligned}&C(a_1+a_2+a_3, \quad a_4+a_5+a_6, \quad a_7+a_8+a_9)\\ \cdot &C(a_1+a_2\gamma+a_3\gamma^2, \; a_4+a_5\gamma+a_6\gamma^2, \; a_7+a_8\gamma+a_9\gamma^2)\\ \cdot &C(a_1+a_2\gamma^2+a_3\gamma, \; a_4+a_5\gamma^2+a_6\gamma, \; a_7+a_8\gamma^2+a_9\gamma).\end{aligned}$$

It should be noted, however, that independently of the said interchange, the identity of these two products of three circulants can be readily shown. As a matter of fact each circulant is resolvable into three linear factors, and the nine linear factors obtained from the one group are the same as those obtained from the other.

17. Similarly from § 14 we learn that the same circulant of circulant arrays, $C_{3.3}$, is equal to

$$C(\phi, \chi, \theta),$$

where

$$\begin{aligned}\phi &= 123+456+789+3(159-168),\\ \chi &= 3(126+459+378),\\ \theta &= 3(129+345+678),\end{aligned}$$

and the column-numbers refer to the array

$$\begin{matrix} a_1 & a_2 & a_3 & a_4 & a_5 & a_6 & a_7 & a_8 & a_9, \\ a_3 & a_1 & a_2 & a_6 & a_4 & a_5 & a_9 & a_7 & a_8, \\ a_2 & a_3 & a_1 & a_5 & a_6 & a_4 & a_8 & a_9 & a_7. \end{matrix}$$

the minors of which are no longer all different—for example, $126=-135=234$, $459=-468=567$, ...

To this there is also a companion form, namely,

$$C(\phi', \chi', \theta'),$$

where ϕ', χ', θ' are outwardly identical with ϕ, χ, θ, but the column-numbers now refer to the array

$$\begin{array}{ccccccccc} a_1 & a_4 & a_7 & a_2 & a_5 & a_8 & a_3 & a_6 & a_9, \\ a_7 & a_1 & a_4 & a_8 & a_2 & a_5 & a_9 & a_3 & a_6, \\ a_4 & a_7 & a_1 & a_5 & a_8 & a_2 & a_6 & a_9 & a_3. \end{array}$$

18. From the preceding paragraph we obtain two rational cubic factors of $C_{3.3}$, namely,

$$\phi + \chi + \theta \quad \text{and} \quad \phi' + \chi' + \theta'.$$

As such, however, these are not new, being essentially the same as those obtained in § 11; so that the resulting equalities

$$\phi + \chi + \theta = C(a_1 + a_2 + a_3, \;\; a_4 + a_5 + a_6, \;\; a_7 + a_8 + a_9),$$
$$\phi' + \chi' + \theta' = C(a_1 + a_4 + a_7, \;\; a_2 + a_5 + a_8, \;\; a_3 + a_6 + a_9),$$

are nothing more than the expression of the change of a three-line determinant with trinomial elements into a sum of 27 determinants with monomial elements. The first-obtained forms, too, have the advantage of showing that the two cubics have a common linear factor, namely, the sum of the a's: so that, up to this point, three rational prime factors of $C_{3.3}$ have been found, one linear and two quadratics. It remains to ascertain the character of the others.

19. To do this it suffices to arrange the nine linear factors of § 16, not in one uninterrupted series, but so as to form a square array, say the array

$$\begin{array}{ccc} F_{11} & F_{12} & F_{13}, \\ F_{21} & F_{22} & F_{23}, \\ F_{31} & F_{32} & F_{33}, \end{array}$$

the positions being chosen so that the F's of the first row are those of

$$C(a_1 + a_2 + a_3, \;\; a_4 + a_5 + a_6, \;\; a_7 + a_8 + a_9),$$

the F's of the first column those of

$$C(a_1 + a_4 + a_7, \;\; a_2 + a_5 + a_8, \;\; a_3 + a_6 + a_9),$$

and therefore F_{11} identical with Σa. The arrangement brings at once to light the existence of two other circulant factors similar to those just mentioned, namely,

$$C(a_1 + a_5 + a_9,\ \ a_2 + a_6 + a_7,\ \ a_3 + a_4 + a_8),$$
$$C(a_1 + a_6 + a_8,\ \ a_2 + a_4 + a_9,\ \ a_3 + a_5 + a_7),$$

these being equal to

$$F_{11} . F_{23} . F_{32},$$
$$F_{11} . F_{22} . F_{33}.$$

We thus learn that $C_{3.3}$ *is resolvable into five rational factors, one linear and four quadratic, the latter being of the form*

$$(x^3 + y^3 + z^3 - 3xyz) \div (x + y + z),$$

i.e.
$$x^2 + y^2 + z^2 - xy - yz - zx.$$

20. After this, one is not surprised to find that there are two other results like (β) of § 11, and therefore also two additional ways of expressing $C_{3.3}$ as the product of three ordinary circulants, namely

$$\begin{aligned} &C\,(a_1 + a_5 + a_9, \quad a_2 + a_6 + a_7, \quad a_3 + a_4 + a_8) \\ &\cdot C\,(a_1 + a_5\gamma + a_9\gamma^2,\ \ a_2 + a_6\gamma + a_7\gamma^2,\ \ a_3 + a_4\gamma + a_8\gamma^2) \\ &\cdot C\,(a_1 + a_5\gamma^2 + a_9\gamma,\ \ a_2 + a_6\gamma^2 + a_7\gamma,\ \ a_3 + a_4\gamma^2 + a_8\gamma), \end{aligned}$$

and

$$\begin{aligned} &C\,(a_1 + a_6 + a_8, \quad a_2 + a_4 + a_9, \quad a_3 + a_5 + a_7) \\ &\cdot C\,(a_1 + a_6\gamma + a_8\gamma^2,\ \ a_2 + a_4\gamma + a_9\gamma^2,\ \ a_3 + a_5\gamma + a_7\gamma^2) \\ &\cdot C\,(a_1 + a_6\gamma^2 + a_8\gamma,\ \ a_2 + a_4\gamma^2 + a_9\gamma,\ \ a_3 + a_5\gamma^2 + a_7\gamma), \end{aligned}$$

Of the two former expressions of this kind (§ 16) one combined the F's of our square array by rows, and the other by columns. In the first of the two just written the sets of three F's forming a circulant are taken from the secondary diagonal and its parallels, and in the second from the main diagonal and its parallels. There are thus twelve three-line circulants that are factors of $C_{3.3}$, and each linear factor has a set of four in each of which it occurs; for example, F_{23} occurs as

$$(a_1 + a_2\gamma + a_3\gamma^2) + (a_4 + a_5\gamma + a_6\gamma^2)\gamma^2 + (a_7 + a_8\gamma + a_9\gamma^2)\gamma,$$
$$(a_1 + a_4\gamma^2 + a_7\gamma) + (a_2 + a_5\gamma^2 + a_8\gamma)\gamma + (a_3 + a_6\gamma^2 + a_9\gamma)\gamma^2,$$
$$(a_1 + a_5 + a_9) \quad + (a_2 + a_6 + a_7)\gamma \quad + (a_3 + a_4 + a_8)\gamma^2,$$

and
$$(a_1 + a_6\gamma + a_8\gamma^2) + (a_2 + a_4\gamma + a_9\gamma^2)\gamma + (a_3 + a_5\gamma + a_7\gamma^2)\gamma^2.$$

21. It will have been observed that the formulæ occurring in these statements of the properties of $C_{3.3}$ are characterized by different groupings of the suffixes of the a's, the four leading groups being

$$\left.\begin{matrix} 1,\ 2,\ 3 \\ 1,\ 4,\ 7 \\ 1,\ 5,\ 9 \\ 1,\ 6,\ 8 \end{matrix}\right\}\ \text{with associates}\ \left\{\begin{matrix} 4,\ 5,\ 6 \\ 2,\ 5,\ 8 \\ 2,\ 6,\ 7 \\ 2,\ 4,\ 9 \end{matrix}\right.\ \text{and}\ \left\{\begin{matrix} 7,\ 8,\ 9, \\ 3,\ 6,\ 9, \\ 3,\ 4,\ 8, \\ 3,\ 5,\ 7. \end{matrix}\right.$$

The first of the four is the originator, and the three others are derivatives of equal status which evolve their associates in one and the same manner, namely, by use of the cyclical changes

$$\begin{matrix} 1,\ 2,\ 3 & \text{into} & 2,\ 3,\ 1, \\ 4,\ 5,\ 6 & \text{into} & 5,\ 6,\ 4, \\ 7,\ 8,\ 9 & \text{into} & 8,\ 9,\ 7. \end{matrix}$$

Now it is very interesting to note that if we return to the general determinant of § 12, which includes $C_{3.3}$ as a special case, a further segregation takes place among the four groups, 1, 4, 7 being removed from its fellowship with 1, 5, 9 and 1, 6, 8. Thus, while it will be found that in regard to the two latter we have the theorem that *the circulant of the arrays of*

$$|a_1 b_2 c_3|,\quad |a_4 b_5 c_6|,\quad |a_7 b_8 c_9|,$$

is equal to the circulant of the arrays

$$\begin{matrix} a_1\,a_5\,a_9 & a_7\,a_2\,a_6 & a_4\,a_8\,a_3 \\ b_7\,b_2\,b_6 & b_4\,b_8\,b_3 & b_1\,b_5\,b_9 \\ c_4\,c_8\,c_3, & c_1\,c_5\,c_9, & c_7\,c_2\,c_6, \end{matrix}$$

and to the circulant of the arrays

$$\begin{matrix} a_1\,a_6\,a_8 & a_4\,a_9\,a_2 & a_7\,a_3\,a_5 \\ b_4\,b_9\,b_2 & b_7\,b_3\,b_5 & b_1\,b_6\,b_8 \\ c_7\,c_3\,c_5, & c_1\,c_6\,c_8, & c_4\,c_9\,c_2, \end{matrix}$$

there is no corresponding theorem in regard to 1, 4, 7, the nearest approach being the theorem of § 10. Again, while we have the theorem of § 12 in regard to 1, 4, 7, there is no such theorem in regard to 1, 5, 9 or 1, 6, 8.

Capetown, S.A.
2nd Dec., 1915.

SOLUTION OF A PROBLEM IN LINEAR DIOPHANTINE APPROXIMATION.

By *W. E. H. Berwick.*

THE theory of continued fractions supplies a solution to the following arithmetical problem: given a positive number α, it is required to find the least value of $|\alpha k - h|$ for integral values of (h, k) with $0 \leq k < N$. In fact if α be expanded in a continued fraction with integral partial quotients, and p/q, p'/q' are the inferior and superior convergents (principal or intermediate) whose denominators are next less than N, $q\alpha - p$ takes a smaller positive value and $q'\alpha - p'$ a greater negative value than $k'\alpha - h'$ for any integral values of h', k' within the above limits for k'.

This theory, however, is insufficient to determine what pair of integers x, y, $0 \leq y < N$, gives its least value to the linear expression

$$f \equiv ax + by + c$$

when a, b, c are real numbers all different from zero. In the following note I have developed a modification of the method of continued fractions, which enables a solution of this latter problem to be given.

In giving a solution of the arithmetical problem it is convenient to make use of geometrical notation and ideas. The geometry is only introduced formally, however, for the sake of visualising the arithmetical processes involved. The solution is a strictly arithmetical one and could be presented logically without mention of any geometrical entity.

On a Cartesian plane, cut up into unit squares by the straight lines

$$x = m, \quad y = n,$$

for integral values of m, n, the distance of a point (m, n) from the line l,

$$ax + by + c = 0,$$

is proportional to $am + bn + c$. So confining attention to the strip of plane lying between $y = 0$, $y = N$ (including the boundary line $y = 0$, but excluding $y = N$ when N is integral), if (m, n) (m', n') are the two integer-points within the strip which are nearest to l, on its positive and negative sides respectively, $am + bn + c$ takes a smaller positive value and $am' + bn' + c$ a greater negative value than $a\mu + b\nu + c$ where (μ, ν) is any other pair of integers with $0 \leq \nu < N$.

It will be convenient to call a point with integral coordi-

nates a *node*, and the straight line joining two nodes a *node-line.* For our purpose then it will be sufficient to show how to find the nearest node to the line l on each side of it within the strip.

The equation of any node-line PQ can be put in the form

$$Ax + By + C = 0,$$

where A, B, C are integers and A, B have no common factor. Every node R then lies on a line

$$Ax + By + C = C'$$

for some integral value of C', and the distance of R from PQ is $\pm C'(A^2+B^2)^{-\frac{1}{2}}$. Further, each of the lines

$$Ax + By + C = \pm 1 \qquad\qquad (1)$$

passes through nodes, and nodes lying on these two lines are equidistant from

$$Ax + By + C = 0,$$

and nearer to it than any other nodes. From this property the two lines (1) are called the node-lines nearest to

$$Ax + By + C = 0.$$

If (ξ, η) is a solution of

$$Ax + By + 1 = 0$$

(obtained by expanding A/B in a continued fraction) all the nodes on the lines

$$Ax + By + C = 1, \quad Ax + By + C = 0, \quad Ax + By + C = -1,$$

have coordinates of the form

$$\left.\begin{array}{ll} \{Bt + (C-1)\,\xi, & -At + (C-1)\,\eta\}, \\ (Bt + C\xi, & -At + C\eta), \\ \{Bt + (C+1)\,\xi, & -At + (C+1)\,\eta\} \end{array}\right\} \qquad (2),$$

respectively for integral values of t.

A plane area is defined by Minkowski to be *convex* when

(1) it is limited in all directions, *i.e.* lies entirely within the rectangle bounded by

$$x = A_1, \quad x = A_2, \quad y = B_1, \quad y = B_2,$$

for finite values of A_1, A_2, B_1, B_2, and

(2) no part of the segment of the straight line joining two arbitrary points on the boundary, and lying between them, falls outside the area.

If Q, R, S are three non-collinear nodes which lie within

a convex area it can be shown that there is at least one node within the area and lying on one of the two node-lines nearest to QR. For let P be that node on the segment QR which is nearer to Q than any other node on it. (If there is no node on QR between Q and R, P coincides with R). The triangle PQS then lies entirely within the convex area. Should this triangle not contain any node other than its vertices, neither can TSQ, T being the fourth vertex of the parallelogram $PQST$, and therefore a node. The entire parallelogram thus contains no node other than its four vertices. Now the whole strip of the plane between PQ and TS can be cut up into parallelograms homothetic to $PQST$ and contiguous to each other, the four vertices of each parallelogram being nodes, and no one of these parallelograms can contain any node except its four corners. There can thus be no node at all between PQ and TS. Hence either S lies on one of the node-lines nearest to PQ, or else there is at least one node P' within PQS or on PS or QS. In the latter case P' lies in the convex area, so repeating the argument for the triangle PQP' it either follows that P' lies on a node-line nearest to PQ or else that there is a node distinct from P' within the triangle or on one of its sides. And since the area PQS, lying within a convex area, can only contain a finite number of nodes, a node in this area and lying on one of the node-lines nearest to PQ must be discovered after a finite number of such steps. We are thus able to state the following theorem:

If P, Q are two nodes within a convex area, and if the area contains any other node at all, then it must contain at least one node on PQ or PQ produced or on one of the two node-lines nearest to PQ.

When the boundary of the area passes through nodes, these nodes, or a part of them, can be included amongst the points belonging to the area if desired, subject only to the restriction that when two points P, Q on the boundary are included amongst points in the area, then every point between P and Q on the straight line PQ is also included amongst them.

A solution of the problem enunciated at the beginning of this note now follows immediately.

First, when it is required to find the nearest node to l, given by

$$f \equiv ax + by + c = 0,$$

between $y = M$ and $y = N$, any two nodes

$$P_1(\alpha_1, \beta_1),\quad P_2(\alpha_2, \beta_2),\quad M < \beta < N,$$

one on each side of the line, are to be chosen.

The parallelogram bounded by

$$\left.\begin{array}{l} y = M, \;\; ax + by + c = a\alpha_1 + b\beta_1 + c \\ y = N, \;\; ax + by + c = a\alpha_2 + b\beta_2 + c \end{array}\right\} \dots\dots\dots (3),$$

(including P_1 and P_2) is then a convex area, and this area contains no point more distant from l than P_1 on one side or P_2 on the other. So if there is any node nearer to l than P_1 or P_2 there must be one such node on P_1P_2, or on one of the node-lines nearest to it. Putting down the coordinates of nodes on these three lines in the form (2) above, that value of t is to be found which corresponds to a node P_3 within the area (3) and gives $|f|$ its least value. Should the only possible values of t belong to P_1, P_2 these are the two nodes nearest to l, one on each side of it, between the limits $M < y < N$, and there is no need for further calculation. But when P_3 is on the same side as P_1, say, the convex area (3) is made narrower on replacing the side

$$ax + by + c = a\alpha_1 + b\beta_1 + c$$

by

$$ax + by + c = a\alpha_3 + b\beta_3 + c,$$

and this narrower area either contains a node P_4 on P_2P_3, or on a node-line nearest to P_2P_3, or else it contains no node at all. In the latter case P_2, P_3 are the nodes nearest to l on its respective sides within the limits considered, and no further calculation is needed. But in the former case P_4 replaces P_2 or P_3 (according as P_4 is on the same side of l as P_2 or P_3), and a further narrowing of the convex area ensues.

Since the original area (3) is limited in all directions it only contains a finite number of nodes, so after a finite number of such steps a stage must be reached when there is no node on P_rP_s or on a node-line nearest to it within the area

$$y = M, \;\; ax + by + c = a\alpha_r + b\beta_r + c,$$

$$y = N, \;\; ax + by + c = a\alpha_s + b\beta_s + c.$$

When such a stage is attained P_r is the nearest node to l on one side of it, and P_s on the other, within the limits $M < y < N$. Of course when l is parallel to a node-line it may happen (and will happen if $M - N$ is great enough) that there are several nodes equidistant from l and nearer to it than any other node on the same side.

When, secondly, it is required to find the node nearest to l on one side of it, the positive side say, two nodes P_1, P_2 are to be chosen on the positive side of l in the strip

$M < y < N$, P_1 being more distant from l than P_2. The first convex area is bounded by

$$\left.\begin{array}{ll} y = M, & ax + by + c = 0, \\ y = N, & ax + by + c = a\alpha_1 + b\beta_1 + c \end{array}\right\} \quad \ldots\ldots\ldots(4),$$

and P_3 is that node in this area which lies on P_1P_2 or on one of the node-lines nearest to it, and is nearer to l than any other such node. P_3 may or may not be nearer to l than P_2, but in either case the convex area (4) is narrowed when the side through P_1 is replaced by a line parallel to it through P_2 or P_3 according as P_3 or P_2 is nearer to l. The next node required in the approximation lies on P_2P_3 or on one of the node-lines nearest to it, and is nearer to l than the more distant of these two points.

Since the area (4) only contains a finite number of nodes a stage must be attained, after a finite number of such steps, when no node on P_mP_n or on a node-line nearest to it within the limits $M < y < N$ is nearer to l than P_n, the nearer of the two points P_m, P_n. When this stage is reached P_n is the nearest node to l within the limits considered and P_m the second nearest.

If it is required to find the third nearest node to l on the positive side within the same limits, the nearest node to

$$ax + by + c - a\alpha_m - b\beta_m - c = 0$$

on its positive side is determined. But it is unnecessary to approximate again from the beginning in finding this third nearest node, only from P_k the nearest of the nodes to l, except P_n and P_m, already discovered. A further approximation enables the fourth nearest node to be found, and the process can be carried to any desired number of steps. It may happen of course that a set of g nodes equidistant from l is found as the s^{th} nearest instead of a single node.

As a numerical example we will find the four nearest nodes to the line

$$f \equiv \pi x - \epsilon y - 1 = 0,$$

$$\pi = 3.141\,592\,653\,590\,\ldots,$$

$$\epsilon = 2.718\,281\,828\,460\,\ldots,$$

on its positive side within the limits $0 \leq y \leq 10\,000$.

The first two nodes being taken to be

$$P_1\,(2,\,0),\ f_1 = 2\pi - 1 = 5.283\,185,$$

$$P_2\,(1,\,0),\ f_2 = \ \pi - 1 = 2.141\,593,$$

P_3 lies on $y=0$ or $y=1$, the other nearest line $y=-1$ being entirely outside the area considered. Now

$$f(t, 0)=\pi t-1=3.141\,593t-1,$$

$$f(t, 1)=\pi t-\epsilon-1=3.141\,593t-3.718\,282,$$

and the least positive values of these two expressions for integral values of t is taken by the second when $t=2$. So

$$P_3 \text{ is } (2, 1), \quad f_3=2.564\,903.$$

The line P_2P_3 is $x-y-1=0$, and P_4 must lie on this or on $x-y=0$ or on $x-y-2=0$. Its coordinates are therefore (t, t) or $(t+1, t)$ or $(t+2, t)$. Now

$$f(t, t)=(\pi-\epsilon)\,t-1=.423\,311t-1,$$

$$f(t+1, t)=(\pi-\epsilon)\,t+\pi-1=.423\,311t+2.141\,593,$$

$$f(t+2, t)=(\pi-\epsilon)\,t+2\pi-1=.423\,311t+5.283\,185.$$

The least positive value of these three expressions is taken by the second when $t=-5$, but this gives a node $(-4, -5)$ outside the limits $0\leq y\leq 10\,000$. Within these limits the least value is taken by the first expression when $t=3$. So

$$P_4 \text{ is } (3, 3), \quad f_4=.269\,932.$$

Continuing the approximation we find

$$
\begin{array}{ll}
P_5\ (4, 4), & f_5=.693\,243, \\
P_6\ (9, 10), & f_6=.091\,516, \\
P_7\ (22, 25), & f_7=.157\,993, \\
P_8\ (41, 47), & f_8=.046\,053, \\
P_9\ (73, 84), & f_9=.000\,590, \\
P_{10}\ (118, 136), & f_{10}=.021\,604, \\
P_{11}\ (240, 277), & f_{11}=.018\,170, \\
P_{12}\ (362, 418), & f_{12}=.014\,736, \\
P_{13}\ (484, 559), & f_{13}=.011\,302, \\
P_{14}\ (606, 700), & f_{14}=.007\,868, \\
P_{15}\ (728, 841), & f_{15}=.004\,434, \\
P_{16}\ (850, 982), & f_{16}=.001\,000, \\
P_{17}\ (5634, 6511), & f_{17}=.000\,023, \\
P_{18}\ (6411, 7409), & f_{18}=.000\,435.
\end{array}
$$

There is no node on $P_{17}P_{18}$, or on a node-line nearest to it, within the parallelogram

$$y=0,\quad y=10\,000,\quad \pi x-\epsilon y-1=0,\quad \pi x-\epsilon y-1=f_{18},$$

so the nearest node to $\pi x-\epsilon y-1=0$ within our limits is (5634, 6511) and the second nearest is (6411, 7409). To find the next one, we find the nearest node to

$$\pi x-\epsilon y-1-f_{18}=0,$$

on its positive side. Among the points already noticed the nearest is P_9 (73, 84). The equation of P_9P_{18} is

$$F\equiv 7325x-6338y-2333=0,$$

and the coordinates of all nodes on

$$F+1=0,\quad F=0,\quad F-1=0,$$

are of the form

$$\begin{aligned}&(6338t+5634, && 7325t+6511),\\ &(6338t+73, && 7325t+84),\\ &(6338t+850, && 7325t+982),\end{aligned}$$

respectively, for integral values of t. Now writing

$$K=6338\pi-7325\epsilon=-.000\,155,$$

we have

$$\begin{aligned}f(6338t+5634,\ 7325t+6511)&=Kt+f_{17}=-.000\,155t+.000\,023,\\ f(6338t+73,\ \ 7325t+84)\ \ &=Kt+f_9\ \,=-.000\,155t+.000\,590,\\ f(6338t+850,\ \ 7325t+982)&=Kt+f_{16}=-.000\,155t+.001\,000,\end{aligned}$$

and within the limits concerned there is no node nearer than that given by $t=0$ in the second of these expressions. Since this point lies on the further boundary of the parallelogram

$$y=0,\quad y=10\,000,\quad \pi x-\epsilon y-1-f_{18}=0,\quad \pi x-\epsilon y-1-f_9=0,$$

it must be the nearest node, except P_{17} and P_{18}, to

$$\pi x-\epsilon y-1=0.$$

Carrying out one more approximation, the nearest node to the line

$$\pi x-\epsilon y-1-f_9=0$$

on its positive side is found to be (7188, 8307), giving

$$7188\pi-8307\epsilon-1=.000\,845.$$

The required nearest node is therefore (5634, 6511), and the next three, in order of distance, are

(6411, 7409), (73, 84) and (7188, 8307).

ON EQUIPOTENTIAL CURVES AS POSSIBLE FREE PATHS.

By *S. Brodetsky, M.A., B.Sc., Ph.D.,* Lecturer in the University of Bristol.

1. THE motion of a particle in a plane is governed by two equations:

$$v^2 = 2V + C \quad\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(1),$$

$$\frac{v^2}{\rho} = \frac{\partial V}{\partial n} \quad\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(2),$$

where v is the velocity, ρ is the radius of curvature of the path, V is the potential of the field in which the motion is taking place, and dn is an element of normal to the path measured in the same direction as the radius of curvature. If an equipotential line is to be a free path, V is the same at all points of the path, so that the velocity is a constant. Equation (2) then gives

$$\frac{1}{\rho} \propto \frac{\partial V}{\partial n}.$$

Let δn represent the normal distance, at any point, from the equipotential in question to a consecutive one. Then

$$\frac{\partial V}{\partial n}\,\delta n = \text{const.}$$

Hence
$$\rho \propto \delta n \quad\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(3).$$

This is the property that must be satisfied if the curves $V=$ const. are to be describable as free paths. An obvious possible case is when the equipotentials are concentric circles, provided the constant V in (1) is properly adjusted.

2. I am not aware that the general solution of (3) has been obtained. To do so in the simplest manner it is convenient to use tangential polar equations. Let p be the perpendicular from the origin on the tangent at any point of an equipotential, and ψ be the inclination of the tangent to the x axis. Then the condition can be written

$$p + \frac{d^2p}{d\psi^2} \propto \delta n \quad\ldots\ldots\ldots\ldots\ldots\ldots(4).$$

If the equipotentials are given by the family

$$p = p(\psi, \lambda),$$

λ being a parameter varying from equipotential to equipotential, then

$$\delta n = \delta p.$$

In going a very short distance along a normal from one equipotential to a consecutive one, ψ remains constant to the first order of small quantities. Hence

$$\delta n = \frac{\partial p}{\partial \lambda} \delta\lambda,$$

and the condition (4) becomes

$$p + \frac{\partial^2 p}{\partial \psi^2} = L(\lambda) \frac{\partial p}{\partial \lambda} \text{................}(5),$$

L being a function of λ only, and λ being invariable in finding the radius of curvature.

Put
$$\mu = \int \frac{d\lambda}{L(\lambda)},$$

and the equation is
$$p + \frac{\partial^2 p}{d\psi^2} = \frac{\partial p}{\partial \mu} \text{......................}(6),$$

and we can suppose μ to be the parameter defining the various equipotentials. The most general solution is

$$p = \Sigma\, e^{(1+h^2)\mu} (A_h e^{h\psi} + B_h e^{-h\psi}) \text{...........}(7),$$

summed for any number of values of h, which can assume any value without restriction. The equation (7) gives us the most general form for the tangential polar equation of a family of equipotential curves which are describable as free paths.

The equation (7) gives for special values of h concentric circles, families of equiangular spirals, cycloids, epi- and hypocycloids, etc.

To plot a curve given by (7) one would first plot as if p and ψ were ordinary polar coordinates, and then construct the first negative pedal.

3. The problem can also be solved by means of pedal coordinates. Let the pedal equation of a family of equipotentials describable as free paths be

$$r^2 = f(\lambda, p) \text{..........................}(8),$$

where λ is a parameter defining the individual members of the family. Then

$$\rho = r\frac{\partial r}{\partial p} = \tfrac{1}{2}\frac{\partial f}{\partial p} \text{......................}(9),$$

the differentiation being partial because λ is constant for any path. Let r', p' refer to a consecutive curve $\lambda+\delta\lambda$. Then

$$r'^2 = f(\lambda+\delta\lambda,\ p').$$

When $\delta\lambda$ is very small we can put

$$\delta n = p' - p = \delta p;$$

and $$r'^2 - r^2 = p'^2 - p^2 = 2p\,\delta p.$$

We get $$2p\,\delta p = f(\lambda+\delta\lambda,\ p+\delta p) - f(\lambda,\ p)$$

$$= \frac{\partial f}{\partial\lambda}\delta\lambda + \frac{\partial f}{\partial p}\delta p.$$

Thus $$\left(2p - \frac{\partial f}{\partial p}\right)\delta p = \frac{\partial f}{\partial\lambda}\delta\lambda.$$

But $$\rho \propto \delta n,\ \textit{i.e.} \propto \delta p.$$

Hence $$\rho = \frac{\frac{1}{2}L(\lambda)\dfrac{\partial f}{\partial\lambda}}{2p - \dfrac{\partial f}{\partial p}}$$

where L is a function of λ, *i.e.* is constant for any path. Substituting in (9) we get for f the partial differential equation

$$\frac{\partial f}{\partial p}\left(\frac{\partial f}{\partial p} - 2p\right) + L(\lambda)\frac{\partial f}{\partial\lambda} = 0 \quad \ldots\ldots\ldots\ldots(10).$$

Put $$-\frac{L(\lambda)}{\dfrac{\partial f}{\partial\lambda}} = \frac{\partial f}{\partial p}\left(\frac{\partial f}{\partial p} - 2p\right) = k^2,$$

where k is an arbitrary constant, and the solution is

$$f = k' - k^2\int\frac{d\lambda}{L(\lambda)} + \int\{p \pm \sqrt{(p^2+k^2)}\}\,dp \ldots\ldots(11),$$

where k' is another arbitrary constant.

Thus we get the paths

$$r^2 - \int\{p \pm \sqrt{(p^2+k^2)}\}\,dp = \text{a constant varying from path to path} \quad \ldots(12),$$

and the radius of curvature at any point is

$$\tfrac{1}{2}\{p \pm \sqrt{(p^2+k^2)}\} \ldots\ldots\ldots\ldots\ldots\ldots(13).$$

This is the complete integral. To obtain the general integral, we put

$$k' = \phi(k^2),$$

where ϕ is an arbitrary functional form. If we eliminate k^2 between the two equations

$$\left.\begin{aligned} r^2 &= \phi(k^2) - k^2\int\frac{d\lambda}{L(\lambda)} + \int\{p \pm \sqrt{(p^2+k^2)}\}\,dp \\ 0 &= \frac{d\phi}{dk^2} - \int\frac{d\lambda}{L(\lambda)} \pm \tfrac{1}{2}\sinh^{-1}\left(\frac{p}{k}\right) \end{aligned}\right\}\ldots(14),$$

we shall get the general solution for r in terms of λ and p.

There is no singular solution.

4. In (14) make

$$\frac{d\phi}{dk^2} = \text{const.};$$

then p/k is a function of λ, and we obtain

$$r^2 = A + M(\lambda)\,p^2 \ldots\ldots\ldots\ldots\ldots\ldots(15),$$

A being a constant, and M a function of λ. $M(\lambda)$ depends upon $L(\lambda)$ in (10), so that it can assume any arbitrary form. $M(\lambda) = 1$ and $A = 0$ give concentric circles. $A = 0$ and $M(\lambda)$ constant give equiangular spirals. If A does not vanish we get epi- and hypocycloids.

The value of ρ in the case of a solution given by the equations (14) is found as follows. In virtue of the equations (14), we have

$$\frac{\partial r^2}{\partial k^2} = 0.$$

Now
$$\rho = \tfrac{1}{2}\left[\frac{\partial r^2}{\partial p}\right],$$

where the brackets denote that we must first substitute for k^2 from the second equation in (14). Hence

$$\left.\begin{aligned} \rho &= \tfrac{1}{2}\frac{\partial r^2}{\partial p} + \tfrac{1}{2}\frac{\partial r^2}{\partial k^2}\frac{\partial k^2}{\partial p} = \tfrac{1}{2}\frac{\partial r^2}{\partial p} \\ &= \tfrac{1}{2}\{p \pm \sqrt{(p^2+k^2)}\} \end{aligned}\right\}\ldots\ldots\ldots(16).$$

The result (16) is identical with (13) obtained for the complete integral. But we must remember that in (16), k^2 is not a constant, but is a function of p and λ defined by the second equation in (14). Thus if

$$\frac{d\phi}{dk^2} = \text{const.}$$

we get p/k as a function of λ, so that

$$\rho = N(\lambda) \cdot p \quad \text{.........................(17)},$$

where $N(\lambda)$ is a function of λ, and is therefore constant for any given path. We thus again get concentric circles, equiangular spirals, cycloids, and epi- and hypocycloids.

5. In the two-dimensional motion we have not considered the forces outside the plane of the path. If the motion is not constrained to be in one plane and there are forces out of the plane, we proceed as follows:

Consider the forces acting upon the particle at any position in its path. Their resultant is along the normal to the equipotential through the position occupied by the particle. Thus the motion of the particle is equivalent to that of a particle on a surface under no forces. The path must therefore be a geodesic on the equipotential surface. For the normal force we have

$$\frac{v^2}{\rho} = \frac{\partial V}{\partial n}$$

as in Art. 1. Thus, as in the two-dimensional problem, we get

$$\rho \propto \delta n \quad \text{.........................(18)}.$$

ρ depends upon the inclination of the geodesic to the lines of curvature of the equipotential surface. Let ρ_1, ρ_2 be the principal radii of curvature at any point on the surface, and let ϕ be the inclination of the geodesic path to the line of curvature corresponding to ρ_1. Then

$$\frac{1}{\rho} = \frac{\cos^2\phi}{\rho_1} + \frac{\sin^2\phi}{\rho_2},$$

and our condition (18) becomes

$$\frac{\cos^2\phi}{\rho_1} + \frac{\sin^2\phi}{\rho_2} \propto \frac{1}{\delta n} \quad \text{.................(19)}.$$

If a line of curvature is also a geodesic, we can put ϕ zero. Such a line of curvature is a plane curve. The problem reduces to the two-dimensional one already investigated. It is in this way that free plane paths along equipotential curves arise.

6. First suppose that the equipotential surfaces are cylinders. One of the principal radii of curvature at any point is infinite, say ρ_2. Then (19) reduces to

$$\rho_1 \sec^2\phi \propto \delta n \quad \text{.........................(20)}.$$

If now the cylindrical equipotentials are such that their section by a plane perpendicular to the generators satisfies the condition for free paths, we have

$$\rho_1 \propto \delta n.$$

But a geodesic on a cylinder is given by

$$\phi = \text{const.}$$

Hence the condition for three-dimensional free paths is also satisfied. We get the result that if a family of cylindrical equipotential surfaces is such that the plane curves obtained by a perpendicular section are describable as free paths, then any loxodrome on any of the cylinders is also freely describable, the velocity being chosen appropriately.

7. We now take the equipotentials to be surfaces of revolution. The meridian curves are geodesics, whilst the parallels of latitude are not geodesics except at points where the tangent plane is parallel to the axis of symmetry. By Clairaut's theorem, the geodesics are given by

$$r \sin\phi = k \qquad (21),$$

where r is the distance of any point from the axis, ϕ is the inclination of the geodesic to the meridian curve, and k is a constant defining a particular geodesic. In the equation (19) we may use for ρ_1 the radius of curvature ρ of the meridian curve, and for ρ_2 we may put $r/\sin\psi$, ψ being the inclination of the normal to the axis of symmetry. Thus our condition becomes

$$\frac{\cos^2\phi}{\rho} + \frac{\sin^2\phi \sin\psi}{r} \propto \frac{1}{\delta n},$$

which may be written

$$\frac{1}{\rho} + \frac{k^2(\rho \sin\psi - r)}{\rho r^3} \propto \frac{1}{\delta n} \qquad (22).$$

If we can find a solution of (22) corresponding to any value of k, then the geodesic $r \sin\phi = k$ will be a possible free path. It is of course obvious that for given k there are an infinite number of equal geodesics on the surface, obtained by imagining one of them to rotate through any angle about the axis of symmetry.

For $k = 0$ the problem reduces to the two-dimensional one already considered, the meridians being the corresponding geodesics.

8. Suppose that we can get two sets of geodesics on the same surface, both being possible free paths, the corresponding constants being k_1 and k_2. We get the two conditions

$$\frac{1}{\rho} + \frac{k_1^2(\rho \sin\psi - r)}{\rho r^3} \propto \frac{1}{\delta n};$$

$$\frac{1}{\rho} + \frac{k_2^2(\rho \sin\psi - r)}{\rho r^3} \propto \frac{1}{\delta n}.$$

Ignoring the trivial case

$$\rho \sin\psi = r,$$

which gives us concentric spheres, on which all great circles are clearly geodesics and possible free paths, we get the two conditions

$$\rho \propto \delta n;$$

$$\frac{r - \rho \sin\psi}{\rho r^3} \propto \frac{1}{\delta n};$$

The latter becomes

$$r - \rho \sin\psi \propto r^3 \ldots\ldots\ldots\ldots\ldots\ldots\ldots (23),$$

the geometric meaning of which is that the distance from the axis of symmetry of the centre of curvature at any point varies as the cube of the distance of this point from the axis of symmetry.

Thus we conclude that for surfaces of revolution, if more than one set of geodesics are described as free paths, the meridians must satisfy the condition for possible free paths in two dimensions, and must also have the property indicated by the condition (23). If this is the case, then it follows that all geodesics are freely describable. It is of course clear that concentric spheres are a family of such surfaces.

It is also not difficult to show that on a surface on which all the geodesics are free paths, the corresponding velocity either continually increases or continually decreases, as the geodesics get more and more inclined to the meridians.

9. If the conditions of the last article are not satisfied, there is still the possibility of one set of geodesics being free paths, if k can be found so that the condition (22) is satisfied.

CRITERIA FOR EXACT DERIVATIVES.

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Prof. Elliott has recently* exhibited a set of criteria of exact derivatives, which differ from the classical system of criteria investigated by Euler, Bertrand, and others. The present paper aims to show how these criteria of Prof. Elliott may be connected with the older criteria, and how certain other sets of criteria may be established and similarly connected.

Reference is made to Prof. Elliott's paper, named above, under the letter $E2$ and to an earlier paper on the subject by the same author† under the letter $E1$.

It should be added that I have not contemplated the presence of more than one dependent variable y, although the results, I believe, admit, in general, of extension to the case of many dependent variables.

§ 1. A convenient notation is the following:

Define

$$O_p^{\,r} \equiv \frac{\partial}{\partial y_r} - pD\frac{\partial}{\partial y_{r+1}} + \frac{p(p+1)}{2\,!}D^2\frac{\partial}{\partial y_{r+2}} - \ldots \text{ to infinity,}$$

the numerical coefficients being those of $(1+x)^{-p}$.

In particular

$$O_p^{\,0} \equiv \frac{\partial}{\partial y} - pD\frac{\partial}{\partial y_1} + \frac{p(p+1)}{2\,!}D^2\frac{\partial}{\partial y_2} \ldots \text{ to infinity,}$$

and $\quad O_0^{\,r} \equiv \dfrac{\partial}{\partial y_r}$.

Such operators obey the two fundamental identities

$$\left.\begin{aligned} D\,.\,O_p^r &\equiv O_{p-1}^{r-1} - O_p^{r-1} \\ \text{and} \qquad O_p^r D &\equiv O_{p-1}^{r-1} \end{aligned}\right\} \ldots\ldots\ldots\ldots\ldots\ldots\ldots (1).$$

In this notation the classical criterion that F be an r^{th} derivative is its annihilation by all of the set $O_1^{\,0}, O_2^{\,1}, \ldots, O_r^{r-1}$.

In addition, I write O_r for yO_r^{r-1}, and ∂, ∂_r for $\dfrac{\partial}{\partial y}, \dfrac{\partial}{\partial y_n}$.

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* *Messenger of Mathematics*, vol. xlv. (1915). † *Loc. cit.*, vol. xliii. (1913).

§ **2.** Introduce the set of operators

$$\left.\begin{aligned}\omega_0 &\equiv y\partial + \ \ y_1\partial_1 + \ \ y_2\partial_2 + \dots \\ \omega_1 &\equiv y\partial_1 + 2y_1\partial_2 + 3y_2\partial_3 + \dots \\ \omega_2 &\equiv y\partial_2 + 3y_1\partial_3 + 6y_2\partial_4 + \dots \\ &\&\text{c.}\quad \&\text{c.}\end{aligned}\right\},$$

the numerical coefficients in ω_n being those in the expansion of $(1-x)^{-(n+1)}$. The operator ω_1 is that called ω by Prof. Elliott in $E2$: ω_0 is of course that due to Euler for homogeneous functions.

These operators are, in point of fact, the operators

$$\frac{\partial}{\partial z},\ \frac{\partial}{\partial z_1},\ \frac{\partial}{\partial z_2},\ \dots,\quad \text{where } z \equiv \log y,\ z_r \equiv \frac{\partial^r z}{\partial x^r}.$$

They are commutable and obey the identities

$$\omega_r D - D\omega_r = \omega_{r-1} \dots\dots\dots\dots\dots\dots\dots (2).$$

In addition we have the results

$$\left.\begin{aligned}O_1 &\equiv \omega_0 - \ \ D\omega_1 + \ \ D^2\omega_2 - \dots \\ O_2 &\equiv \omega_1 - 2D\omega_2 + 3D^2\omega_3 - \dots \\ O_3 &\equiv \omega_2 - 3D\omega_3 + 6D^2\omega_4 - \dots \\ &\&\text{c.}\quad \&\text{c.}\end{aligned}\right\} \dots\dots\dots\dots (3).$$

(These latter identities represent, of course, merely the transformation of the operators O_r^{r-1} from variables $y, y_1, y_2, \dots$, to variables $z, z_1, z_2, \dots$).

Now the annihilator of r^{th} derivatives given by Prof. Elliott in his recent paper ($E2$) is the operator

$$(r, w) \equiv (D\omega - ri)\{D\omega - (r+1)i\} \dots (D\omega - wi),$$

where w is the greatest weight of the function in $y_1, y_2, y_3, \dots$.

From the identities (3) we have the set

$$\left.\begin{aligned}O_1 &= \ \ \omega_0 - D\omega_1 + D^2(\dots) \\ 2O_1 + DO_2 &= 2\omega_0 - D\omega_1 + D^3(\dots) \\ 3O_1 + 2DO_2 + D^2O_3 &= 3\omega_0 - D\omega_1 + D^4(\dots) \\ &\&\text{c.}\ .\ \&\text{c.}\end{aligned}\right\}.$$

Hence $$(2, w)\, O_1 = (2, w)(\omega_0 - D\omega_1),$$

since $$(2, w)\, D^2 \equiv 0,$$

and since the operators on the right are of zero weight. But on a function homogeneous of degree i, $\omega_0 \equiv i$.

Thus $$(2, w)\, O_1 = -(2, w)\,(D\omega - i) = -(1, w),$$
so $$(3, w)\,(2O_1 + DO_2) = -(3, w)\,(D\omega - 2i) = -(2, w),$$
&c. &c.

And so on: finally

$$(w, w)\,\{(w-1)\, O_1 + (w-2)\, DO_2 + \ldots\} = -(w-1, w)$$
$$wO_1 + (w-1)\, DO_2 + \ldots = wi - D\omega$$
$$= -(w, w).$$

This gives

$$\left.\begin{aligned} (-)^w (1, w) &= \{wO_1 + (w-1)\, DO_2 + \ldots\} \\ &\qquad \{(w-1)\, O_1 + (w-2)\, DO_2 + \ldots\} \ldots O_1 \\ (-)^{w-1}(2, w) &= \{wO_1 + (w-1)\, DO_2 + \ldots\} \ldots (2O_1 + DO_2) \\ &\text{\&c. \&c.} \end{aligned}\right\} \ldots (4),$$

expressing the operators (r, w) in terms of the operators O_r, and showing that a function annihilated by all of $O_1, O_2, \ldots, O_r$ is annihilated by all of $(1, w)$, $(2, w)$, … (r, w).

§ 3. We may further deduce from the equations (3) the identities

$$\left.\begin{aligned} O_1 &= i - D\,(\omega_1 - D\omega_2 + D^2\omega_3 - \ldots) \\ O_1 + DO_2 &= i - D^2\,(\omega_2 - 2D\omega_3 + 3D^2\omega_4 - \ldots) \\ O_1 + DO_2 + D^2 O_3 &= i - D^3\,(\omega_3 - 3D\omega_4 + 6D^2\omega_5 - \ldots) \\ &\text{\&c. \&c.} \end{aligned}\right\} \ldots\ldots (5).$$

The function is supposed homogeneous of degree i.

From these identities we see that annihilation by the *one* operator $P_r \equiv O_1 + DO_2 + D^2 O_3 + \ldots D^{r-1} O_r$ is (if the function be homogeneous of degree other than zero) sufficient to prove it an r^{th} derivative $D^r\phi$, and that $i\phi$ can be expressed as

$$\omega_r - rD\omega_{r+1} + \frac{r(r+1)}{2} D^2\omega_{r+2} - \ldots .$$

The fact that annihilation by P_r necessitates annihilation by $P_{r-1}, P_{r-2}, \ldots, P_1$ can be exhibited by means of the identities

$$\left.\begin{aligned} i^{r-1} P_1 &= P_1 P_2 \ldots P_r \\ i^{r-2} P_2 &= P_2 \ldots P_r \\ &\text{\&c. \&c.} \end{aligned}\right\} \ldots\ldots\ldots\ldots\ldots\ldots (6).$$

§4. Now we have expressed the annihilators $(1, w)$, $(2, w)$ ... (r, w) in terms of the annihilators O_1, O_2, ... O_r, and shown that the latter necessitate the former. Conversely, although it does not seem possible to express O_r in terms of $(1, w)$, $(2, w)$ ... (r, w), we can show that annihilation by (r, w) necessitates annihilation by O_1, ..., O_r.

For this we employ the identity of $E2$ (§§ 5, 6), namely, that

$$1 \equiv A_0(1, w) + A_1 D\omega\,(2, w) + A_2 D^2\omega^2\,(3, w) + \ldots\ldots (7),$$

where A_0, A_1, A_2 ... are numerical quantities: precisely

$$A_r = (-1)^{w-r} i^{-w} \frac{1}{r!\,(w-r)!}.$$

It may be first mentioned that the set of identities

$$O_r\omega - \omega O_r = r i O_{r+1}$$

holds, whence $\qquad \omega O_r = O_{r-1}(D\omega - ri)$;

in addition we see that

$$O_1,\; O_1\omega,\; O_1\omega^2,\; \ldots,\; O_1\omega^{r-1}$$

form a set of annihilators equivalent to O_1, O_2, ..., O_r.

Operating on both sides of the identity (7) with O_1 we have, since $O_1 D = 0$,

$$O_1 = A_0\, O_1(1, w).$$

Operating with O_2 we have, since $O_2 D = O_1$ and $O_2 D^2 = 0$,

$$\begin{aligned} O_2 &= A_0 O_2 (1, w) + A_1 O_1 \omega\,(2, w) \\ &= \{A_0 O_2 (D\omega - 1) + A_1 O_1 \omega\}\,(2, w) \\ &= (A_0 \omega O_1 + A_1 O_1 \omega)\,(2, w). \end{aligned}$$

Similarly $\qquad O_3 = (A_0\omega^2 O_1 + A_1 \omega O_1 \omega + A_2 O_1 \omega^2)\,(3, w)$,

and so forth.

It is clear then that annihilation by (r, w) necessitates annihilation by O_1, O_2, ..., O_r.

§5. I pass now to the case in which the functions dealt with are isobaric in y_1, y_2, ..., y_n. Since, when x occurs explicitly,

$$D \equiv \frac{\partial}{\partial x} + y_1\partial + y_2\partial_1 + \ldots,$$

i.e. is not isobaric, a function and its derivative will not, with rare exceptions, be isobaric, and we therefore stipulate, for

isobaric functions, that x be not present explicitly, and therefore accurately

$$D \equiv y_1 \partial + y_2 \partial_1 + y_3 \partial_2 + \dots .$$

In this case we can introduce a new set of annihilators E_r, defined as follows:—

$$1 + E_r \equiv D^{r-1} (y_1 O_r^r + y_2 O_r^{r+1} + y_3 O_r^{r+2} + \dots).$$

To prove them to be annihilators we proceed thus

$$\begin{aligned}(1 + E_r) D &= D^{r-1} (y_1 O_r^r D + y_2 O_r^{r+1} D + \dots) \\ &= D^{r-1} (y_1 O_{r-1}^{r-1} + y_2 O_{r-1}^r + \dots) \\ &= D (1 + E_{r-1}).\end{aligned}$$

Thus
$$E_r D = D E_{r-1}.$$

In particular

$$\begin{aligned}(1 + E_1) D &= y_1 O_0^0 + y_2 O_0^1 + y_3 O_0^2 + \dots \\ &= y_1 \partial + y_2 \partial_1 + y_3 \partial_2 + \dots \\ &= D,\end{aligned}$$

so that $\quad E_1 D = 0 \quad$ and $\quad E_r D^r = D^{r-1} E_1 D = 0.$

Thus E_r is an annihilator of r^{th} derivatives.

§ 6. To prove the converse we observe in the first place that a function F annihilated by E_r satisfies the equation

$$F = D^{r-1} (y_1 O_r^r + y_2 O_r^{r+1} + \dots) F,$$

and thus is certainly an $(r-1)^{\text{th}}$ derivative. It is therefore annihilated by all of $E_{r-1}, E_{r-2}, \dots, E_1$. We may exhibit this fact, that annihilation by E_r necessitates annihilation by all the E's of lower suffix, by means of identities of the type

$$E_r . E_{r+s} \equiv E_r .$$

It is to be noted that we have also proved that, for a function annihilated by E_r, $(r-1)$ integrations can be performed by direct differential operation, and this without knowledge that the function is either isobaric or homogeneous.

Now, employing the identities (1) of § 1, we have

$$\begin{aligned}&D (y_1 O_r^r + y_2 O_r^{r+1} + y_3 O_r^{r+2} + \dots) \\ &= y_2 O_r^r + y_3 O_r^{r+1} + y_4 O_r^{r+2} + \dots) \\ &+ y_1 (O_{r-1}^{r-1} - O_r^{r-1}) + y_2 (O_{r-1}^r - O_r^r) + \dots \\ &= y_1 O_{r-1}^{r-1} + y_2 O_{r-1}^r + \dots - y_1 O_r^{r-1}.\end{aligned}$$

Hence $$E_r + 1 = E_{r-1} + 1 - D^{r-2} y_1 O_r^{r-1},$$

i.e. $$\left.\begin{aligned} E_{r-1} - E_r &= D^{r-2} y_1 O_r^{r-1}, \\ DE_1 &= -y_1 O_1^0 \end{aligned}\right\}.$$

and in particular

Thus the E's are connected with the O's, and if $E_r F = 0$, since we know that $E_{r-1}F$ must also be zero, it follows that

$$D^{r-2} y_1 O_r^{r-1} F = 0.$$

If x is not to occur explicitly, this must lead in all but a few exceptional cases to $O_r^{r-1} F = 0$, which proves F an r^{th} derivative.

We see then that functions annihilated by E_r are r^{th} derivatives.

The properties of the particular operator E_1 have been previously described by Prof. Elliott (E1, part I).

§ 7. There is finally a system of annihilators analogous to the system discussed by Prof. Elliott in E2, and mentioned above under the symbol (r, w).

We introduce these as follows:

Define the set of operators (analogous to the set ω_r)

$$\left.\begin{aligned} \eta_0 &\equiv y_1 \partial_1 + 2y_2 \partial_2 + 3y_3 \partial_3 + \dots \\ \eta_1 &\equiv y_1 \partial_2 + 3y_2 \partial_3 + 6y_3 \partial_4 + \dots \\ \eta_2 &\equiv y_1 \partial_3 + 4y_2 \partial_4 + 10y_3 \partial_5 + \dots \\ &\&c. \quad \&c. \end{aligned}\right\} \dots\dots\dots\dots (8).$$

Writing η for η_1, we have $\eta D - D\eta = \eta_0$; but η_0 operating on a function isobaric of weight w multiplies that function by w.

By the symbol $[0, w]$ mean the operator

$$D\eta\,\{D\eta - (w-1)\}\,\{D\eta - (2w-3)\}\,\{D\eta - (3w-6\}\dots\left\{D\eta - \frac{w(w-1)}{2}\right\},$$

where the r^{th} factor from the beginning is $D\eta - (r-1)(w - \tfrac{1}{2}r)$ and there are w factors in all.

Mean by $[1, w]$ the operator obtained from the foregoing by removing the first factor, *i.e.* it starts with $D\eta - (w-1)$: mean by $[2, w]$ the operator obtained by removing the first two factors of $[0, w]$, and so forth.

Then we shall prove that $[0, w]$ annihilates all integral algebraic functions isobaric of weight w; $[1, w]$ annihilates all integral algebraic first derivatives of weight w; and generally $[r, w]$ annihilates all integral algebraic r^{th} derivatives of weight w.

To prove this observe that $[0, w]$ may be written

$$D\left\{\eta D-\frac{w(w-1)}{2}\right\}\ldots\{\eta D-(2w-3)\}\{\eta D-(w-1)\}\eta.$$

But if F is of weight w, ηF is of weight $(w-1)$: the factors $\eta D-(w-1)$, $\eta D-(2w-3)$, &c., are of zero weight. Thus in the above operator we may write $\eta D=D\eta+(w-1)$.

This reduces it to

$$D\left\{D\eta-\frac{(w-1)(w-2)}{2}\right\}\ldots\{D\eta-(w-2)\}D\eta.\eta.$$

In other words

$$[0, w]F_w=D[0, w-1]\eta F_w;$$

since ηF_w is of weight $w-1$,

$$[0, w-1]\eta F_w=D[0, w-2]\eta^2 F_w.$$

Proceeding in this way we see that

$$[0, w]F_w=D^{w-1}[0, 1]\eta^{w-1}F_w.$$

But, the weight of $\eta^{w-1}F_w$ being unity, it can involve only y and y_1, and is thus annihilated by $[0, 1]$, which is $D\eta$.

Again $[1, w]D$ may be written

$$D\{\eta D-(w-1)\}\{\eta D-(2w-3)\}\ldots\left\{\eta D-\frac{w(w-1)}{2}\right\}.$$

Now if $F(\equiv D\phi)$ is of weight w, ϕ is of weight $w-1$, so that

$$[1, w]D\phi=D\{D\eta\}\{D\eta-(w-2)\}\ldots\left\{D\eta-\frac{(w-1)(w-2)}{2}\right\}\phi$$
$$=D[0, w-1]\phi=0,$$

by the preceding result.

In like manner we may prove that

$$[2, w]D^2\phi_{w-2}=D[1, w-1]D\phi_{w-2}=0,$$

and proceed similarly to show that $[r, w]$ annihilates r^{th} derivatives.

§ 8. To prove the converse, that functions annihilated by $[r, w]$ are r^{th} derivatives, we establish the identity

$$1=A_0[1,w]+A_1 D\eta[2,w]+A_2 D\eta\{D\eta-(w-1)\}[3,w]+\ldots \quad (9).$$

where A_0, A_1, A_2, ... are the numerical quantities which secure the partial-fraction identity

$$\frac{A_0}{x}+\frac{A_1}{x-(w-1)}+\frac{A_2}{x-(2w-3)}+\dots$$
$$\equiv\frac{1}{x\{x-(w-1)\}\dots\left\{x-\frac{w(w-1)}{2}\right\}}.$$

Writing the identity (9) in the form

$$1=A_0[1,w]+A_1D\eta[2,w]+A_2D^2\eta^2[3,w]+A_3D^3\eta^3[4,w]+\dots(9'),$$

we see that a function F annihilated by $[1, w]$ is a first derivative and that its integral is

$$\{A_1\eta[2, w]+A_2D\eta^2[3, w]+\dots\}F.$$

Since $[2, w]$ is a factor of $[1, w]$, annihilation by $[2, w]$ of a function F shows that F is a second derivative $D^2\phi$, and allows us to write down ϕ by direct differential operation only. So generally for $[r, w]$.

§9. It remains to connect this set of annihilators with the foregoing systems. To do this we need the operators η_r defined above and the following identities that may be obtained, expressing the operators E_r in terms of these:

$$\left.\begin{aligned}
E_1&=(\eta_0-1)-D\eta_1+D^2\eta_2-D^3\eta_3+\dots\\
E_2&=\quad -1\quad +D\eta_1-2D^2\eta_2+3D^3\eta_3-\dots\\
E_3&=\quad -1\quad +D^2\eta_2-3D^3\eta_3+\dots\\
E_4&=\quad -1\quad +D^3\eta_3-\dots\\
&\&c.\quad \&c.
\end{aligned}\right\}\dots(10).$$

From these we obtain the set

$$\left.\begin{aligned}
E_1&=(\eta_0-1)-D\{\eta_1-D\eta_2+D^2\eta_3-\dots\}\\
E_1+E_2&=(\eta_0-2)-D^2\{\eta_2-2D\eta_3+3D^2\eta_4-\dots\}\\
E_1+E_2+E_3&=(\eta_0-3)-D^3\{\eta_3-3D\eta_4+6D^2\eta_5-\dots\}\\
&\&c.\quad \&c.
\end{aligned}\right\}.$$

Since a function annihilated by E_r is annihilated by all of $E_1, E_2, \dots, E_{r-1}$, we have here an additional proof of the fact that a function annihilated by E_r (if it is isobaric of weight

other than r) is an r^{th} derivative $D^r\phi$: moreover we see that we have an expression for $(w-r)\,\phi$, namely

$$\eta_r - rD\eta_{r+1} + \frac{r(r+1)}{2!}D^2\eta_{r+2} - \dots.$$

§ 10. Another set of identities that can be deduced from the set (10) is

$$\left.\begin{aligned} E_1 &= (\eta_0 - 1) - D\eta + D^2(\dots) \\ 2E_1 + E_2 &= (2\eta_0 - 3) - D\eta + D^3(\dots) \\ 3E_1 + 2E_2 + E_3 &= (3\eta_0 - 6) - D\eta + D^4(\dots) \\ &\text{\&c.} \quad \text{\&c.} \end{aligned}\right\}$$

Since $$[r, w]\,D^r \equiv 0,$$
we have $$[2, w]\,E_1 = [2, w]\,\{(w-1) - D\eta\} = -[1, w],$$
so $$[3, w]\,(2E_1 + E_2) = -[2, w],$$
and finally
$$[w-1, w]\,\{(w-2)\,E_1 + (w-3)\,E_2 + \dots\} = -[w-2, w]$$
$$\{(w-1)\,E_1 + (w-2)\,E_2 + \dots\} = -[w-1, w].$$
Hence
$$\left.\begin{aligned} [1, w] &= (-)^{w-1}\{(w-1)\,E_1 + \dots\}\{(w-2)\,E_1 + \dots\}\dots\{2E_1 + E_2\}\,E_1 \\ [2, w] &= (-)^{w-1}\{(w-1)\,E_1 + \dots\}\{(w-2)\,E_1 + \dots\}\dots\{2E_1 + E_2\} \\ &\text{\&c.} \quad \text{\&c.} \end{aligned}\right\}.$$

Thus the operators $[r, w]$ are expressed in terms of the operators E_r, and it is clear that annihilation by E_r necessitates annihilation by $[r, w]$. To prove the converse we employ the identity (9′) and proceed exactly as in the case of the annihilators O_r and ω_r.

§ 11. It may be remarked that it follows from the foregoing results, in conjunction with previously known facts, that a function (separable into isobaric and homogeneous parts), not involving x explicitly and known to be an r^{th} derivative, can always have its r integrations performed by direct differential operation, except in the case when, with one differentiation left to be performed, the function is of unit weight and zero degree—that is, of course, the case covering the possibility of the original function being logarithmic.

AN ARITHMETICAL PROOF OF A CLASS RELATION FORMULA.

By *L. J. Mordell*, Birkbeck College, London.

LET $F(m)$ be the number of uneven classes of negative determinant $-m$, with the convention that the class $(1, 0, 1)$ and its derived classes are each reckoned as $\frac{1}{2}$, and that $F(0)=0$. It is well known that

$$F(m)-2F(m-1^2)+2F(m-2^2)-\ldots=-\Sigma(-1)^{\frac{1}{2}(a+d)}d\ldots(A),$$

where the left-hand side is continued so long as the argument of the function F is not negative; and the right-hand summation refers to all the divisors d of m, which are $\leq\sqrt{(m)}$ and of the same parity as their conjugate divisors a, but when $d=\sqrt{(m)}$, the coefficient d in the sum is replaced by $\frac{1}{2}d$.

Kronecker* proved this formula and similar ones by considering in the theory of elliptic functions the modules which admit of complex multiplication. Hermite* shewed that formulae of this kind could be proved by expanding in different ways functions represented by products and quotients of theta functions, although when a formula is given it is no easy matter to see *a priori* what is the function to be expanded. This method was not unknown to Kronecker.* Liouville* showed that the general formulæ introduced by him in the *Theory of Numbers* could be applied to give an arithmetical proof of some formulæ of this kind. Kronecker† also gave an arithmetical proof depending upon the general theory of bilinear forms with four variables.

By considering the subject from rather a different point of view, I was enabled to find directly various formulae of this kind, some of which are given in my "Note on Class Relation Formulæ.‡ But some of my analytical methods suggest a very easy arithmetical transformation, and as an illustration, I prove the above formula.

Let m be any given positive integer (all the letters used denote integers), and consider the representations of m by the two forms

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* An account of these methods will be found in H. J. S. Smith, *Report on Theory of Numbers*, § 6, Collected Works, vol. i., pp. 322–350.

† Collected Works, vol. ii., p. 427. *Ueber Bilineare Formen mit vier variabeln.*

‡ *Messenger of Mathematics*, 1915, vol. xlv., pp. 76–80. Mr. G. Humbert writes to me that the formula (A) in this paper has been given by him in his paper, "Nombre de classes des formes quadratiques," in Liouville, 1907; and that it is due to K. Petr, *Acad. des Sciences de Bohême*, 1900–1901. Both of these authors have found this formula and various others, some involving the representation of numbers by simple indefinite forms, by means of Hermite's classical method.

$$s^2 + n^2 + n(2t+1) - r^2 = m \quad \ldots\ldots\ldots\ldots (1),$$

$$d(d+2\delta) = m \quad \ldots\ldots\ldots\ldots (2),$$

in which s takes all values, positive, negative, and zero; n all positive values, zero excluded; r all positive, negative, and zero values from $-(n-1)$ to n, both included; and t all positive values, zero included. d and δ are positive, but δ may also take the value zero. Let $f(x)$ be any even function (either an analytic function or an arithmetic function) of x, so that $f(x) = f(-x)$. Then

$$\Sigma (-1)^r f(r+s) = -2\Sigma (-1)^\delta d f(d) \quad \ldots\ldots\ldots (B),$$

where the summation on the left extends to all solutions of equation (1), and the summation on the right to all solutions of equation (2); but when $\delta = 0$, the coefficient 2 in the sum is replaced by unity.

For putting $r + s = \pm k$, and supposing k a given positive integer, the coefficient of $f(k)$ on the left-hand side is $\Sigma(-1)^r$ extended to all solutions of

$$k^2 \pm 2kr + n^2 + n(2t+1) = m \quad \ldots\ldots\ldots\ldots (3),$$

where t, n and r are limited as in equation (1). But this coefficient is equal to $2\Sigma(-1)^{n+\sigma}$, extended to all solutions of

$$k^2 + n^2 + n\xi + 2k\sigma = m \ldots\ldots\ldots\ldots\ldots \quad (4),$$

where n is as before, ξ takes all positive and negative odd values from $-(2k-1)$ to $2k-1$, both included, σ all positive values, zero included, but when $\sigma = 0$, $2(-1)^{n+\sigma}$ is replaced by $(-1)^{n+\sigma}$.

The proof of this depends upon the theorem* that if n and k are given positive integers, $\Sigma(-1)^r$ extended to all solutions of

$$n(2t+1) \pm 2kr = N \quad \ldots\ldots\ldots\ldots\ldots\ldots (5),$$

in which r and t are limited as above, is equal to $2\Sigma(-1)^{n+\sigma}$, extended to all solutions of

$$n\xi + 2k\sigma = N \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots (6),$$

in which ξ, σ are limited as above, and with the above convention when $\sigma = 0$. Putting $\xi = 2\eta + 1$ and $N - n = 2P$, where we may suppose P is an integer, otherwise both equations have no solution, and the theorem is certainly true, this is the same as $\Sigma(-1)^r$ extended to all solutions of

$$nt \pm kr = P \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (7),$$

* It was suggested by evaluating $\sum_{r=n}^{-(n-1)} \sum_{t=0}^{\infty} q^{n(2t+1)\pm 2kr}$.

is equal to $2\Sigma(-1)^{\eta+\sigma}$ extended to all solutions of

$$n\eta + k\sigma = P \quad\ldots\ldots\ldots\ldots\ldots\ldots(8),$$

where $\eta = 0, \pm 1, \ldots, \pm(k-1), -k$, and σ, t, r are limited as above.

Putting $-r$ for r in the equation with the negative sign in (7), this is the same as $\Sigma(-1)^r$ extended to all solutions of $nt + kr = P$ where r takes the values $-n$ to $n-1$ and then again the values $n-1$ to $-n$ plus $(-1)^r$ for the solution $r = n$ [if $r = n$ does not give a positive integer value for t, $(-1)^r$ must be replaced by zero, and similarly in other cases] minus $(-1)^r$ for the solution $r = -n$ equals $2\Sigma(-1)^{\eta+\sigma}$ extended to all solutions of $n\eta + k\sigma = P$ where $\eta = 0, \pm 1, \ldots, \pm(k-1), -k$ minus $(-1)^{\eta+\sigma}$ for the solution $\sigma = 0$, where now the convention for $\sigma = 0$ is removed. But $(-1)^r$ for $r = n$ minus $(-1)^r$ for $r = -n$ is equal to $-(-1)^{\eta+\sigma}$ for $\sigma = 0$. For we may consider P/n to be an integer, as otherwise none of these solutions exist. If $k > P/n \geq -k$, there is a solution $\sigma = 0$, but the case $r = n$ does not arise while the case $r = -n$ does and the equality is clear. If P/n lies outside these limits, the cases $r = n$ and $r = -n$ both arise and $(-1)^n$ cancels $(-1)^{-n}$, while the solution $\sigma = 0$ does not arise, so that again the equality is evident.

Putting now $r - n$ for r and $\eta - n$ for η, we have to show $\Sigma(-1)^r$ for all solutions of

$$nt + kr = Q \quad\ldots\ldots\ldots\ldots\ldots\ldots(9),$$

$r = 0, 1, 2, \ldots, 2n-1$ equals $\Sigma(-1)^\sigma$ for the solutions of

$$n\eta + k\sigma = Q \quad\ldots\ldots\ldots\ldots\ldots\ldots(10),$$
$$\eta = 0, 1, 2, \ldots, 2k-1.$$

But we can establish a unique correspondence between the solutions of equations (9) and (10). Thus if (10) admits of a solution (η, σ), we may suppose $\eta < k$, for otherwise $\eta - k$, $\sigma + n$ is such a solution. Hence we can arrange its solutions as follows (η, σ) with $0 \leq \sigma < n$, and in pairs such as (η, σ) and $(\eta + k, \sigma - n)$ with $\sigma \geq n$. If $\sigma < n$ we can take $r = \sigma$ and $t = \eta$. If $\sigma \geq n$, we write $r = \sigma + en$, $t = \eta - ek$, and there are two consecutive values of e for which we can make $0 \leq r < 2n$ and these make $t \geq 0$. Since for the pairs of solution of (10) for which (if $a \geq 1$)

$$\{0 \leq \eta < k,\ an \leq \sigma < (a+1)\,n\} \text{ or } \{k \leq \eta \leq 2k,\ (a-1)\,n \leq \sigma < an\}$$

we find

$$\{0 \leq r < n,\ ak \leq t < (a+1)\,k\} \text{ or } \{n \leq r < 2n,\ (a-1)\,k \leq t < ak\},$$

the correspondence is obviously unique. But

$$(-1)^{\sigma}+(-1)^{\sigma-n}=(-1)^{\sigma+en}+(-1)^{\sigma+(e+1)n}$$

which proves the statement for equations (9) and (10); so that we can replace equation (3) by equation (4).

Hence $\Sigma(-1)^r$ extended to all solutions of equation (3) is equal to $2\Sigma(-1)^{n+\sigma}$ extended to all solutions of

$$k^2+n^2-n\xi+2k\sigma=m,$$

for which n takes all positive and negative values, zero excluded; ξ takes the values 1, 3, ..., $(2k-1)$, $\sigma=0$, 1, 2, ... with the convention for $\sigma=0$.

Noticing now that we can group the solutions in pairs, such as n, ξ, σ and $\xi-n$, ξ, σ, if ξ is not equal to n, then since ξ is odd the sum $2\Sigma(-1)^{n+\sigma}$ is zero for this pair of solutions, and we need only consider those solutions for which $n=\xi$. But then $k(k+2\sigma)=m$, and the sum reduces to $\Sigma(-1)^{\xi+\sigma}$, which being summed for $\xi=1, 3, \ldots, (2k-1)$ gives $-2k(-1)^{\sigma}$. This proves the result (B).

Taking now $f(x)=(-1)^x$, we have $\Sigma(-1)^s$, extended to all solutions of equation (1) is equal to $-2\Sigma(-1)^{\delta+d}d$ extended to all solutions of equation (2). But when s is given it can be shown that the number of solutions of (1) is $2F(m-s^2)$. This I have done in my forthcoming paper, "On Class Relation Formulæ." It can also be proved very simply by means of the modular division of the plane, a method due to Humbert (*l. c.*) Hence

$$\Sigma(-1)^s F(m-s^2)=-\Sigma(-1)^{\delta+d}d,$$

or $\quad F(m)-2F(m-1^2)+2F(m-2^2)\ldots=-\Sigma(-1)^{\frac{1}{2}(a+d)}d.$

NOTE ON THE ZEROS OF RIEMANN'S ζ-FUNCTION.

By *J. R. Wilton, M.A., D.Sc.*

THE following slight extension of Mr. Hardy's result that $\zeta(s)$ has an infinite number of roots on the line $\sigma=\frac{1}{2}$ may not be without interest. It is here shown that both the real and the imaginary parts of $\Gamma(\frac{1}{2}s)\,\pi^{-\frac{1}{2}s}\,\zeta(s)$ have an infinite number of roots on any line $\sigma=\sigma_0$, such that $0<\sigma_0<1$. The method followed, except in the actual determination of the value of the definite integral which leads to the result, is the same as that adopted by Mr. Hardy.*

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* *Comptes Rendus*, April, 1914, pp. 1012–4.

Riemann's integral for $\zeta(s)$ is

$$\Gamma(\tfrac{1}{2}s)\,\pi^{-\frac{1}{2}s}\zeta(s) = \frac{1}{s(s-1)} + \int_1^\infty \left(\mu^{\frac{1}{2}(1-s)} + \mu^{\frac{1}{2}s}\right) \sum_{n=1}^{\infty} e^{-\pi n^2 \mu} \frac{d\mu}{\mu}.$$

In this put

$$s = \tfrac{1}{2}(1 + \iota z) = \tfrac{1}{2}(1 - y + \iota x),$$

and $\mu = e^{4\lambda}$; we obtain*

$$\phi(x, y) + \iota\psi(x, y) \equiv F(z) \equiv -\tfrac{1}{4}\Gamma\{\tfrac{1}{4}(1+\iota z)\}\pi^{-\frac{1}{4}(1+\iota z)}\zeta\{\tfrac{1}{2}(1+\iota z)\}$$

$$= \frac{1}{1+z^2} - 2\int_0^\infty \cos z\lambda \,.\, e^\lambda \sum_{n=1}^{\infty} e^{-\pi n^2 e^{4\lambda}} d\lambda$$

$$= \int_0^\infty \cos z\lambda \,.\, f(\lambda)\, d\lambda \quad \text{......(1)},$$

where $\quad f(\lambda) = e^{-\lambda} - 2e^\lambda \sum_{n=1}^{\infty} e^{-\pi n^2 e^{4\lambda}},$

provided that $\quad -1 < y < 1 \quad$ ......(2),

which is equivalent to $0 < \sigma < 1$.

Further, from Jacobi's relation

$$1 + 2\sum_{n=1}^{\infty} e^{-\pi n^2/\mu} = \mu^{\frac{1}{2}}\left(1 + 2\sum_{n=1}^{\infty} e^{-\pi n^2 \mu}\right),$$

we obtain, on putting $\mu = e^{4\lambda}$, the relation $f(\lambda) = f(-\lambda)$, so that of the two functions

$$\theta(\lambda) = \cosh y\lambda \,.\, f(\lambda), \quad \chi(\lambda) = \sinh y\lambda \,.\, f(\lambda) \quad \text{......(3)},$$

θ is even and χ is odd, and on account of (2) both vanish together with all their differential coefficients at infinity. Also $\theta^{(2n-1)}(0) = 0$ and $\chi^{(2n)}(0) = 0$.

From (1) and (3) we have, by successive integration by parts,

$$2x^{2n}\phi(x, y) = x^{2n}\int_{-\infty}^{\infty} \cos x\lambda \,.\, \theta(\lambda)\, d\lambda,$$

$$= (-)^n \int_{-\infty}^{\infty} \cos x\lambda \,.\, \theta^{(2n)}(\lambda)\, d\lambda,$$

$$0 = \int_{-\infty}^{\infty} \sin x\lambda \,.\, \theta^{(2n)}(\lambda)\, d\lambda;$$

* In Riemann's notation

$$\xi(t) \equiv \tfrac{1}{2}(1+4t^2)F(2t) = \int_0^\infty \cos 2\lambda t \,.\left(\frac{d^2}{d\lambda^2}\right) - 1\left\{e^\lambda \sum_{n=1}^{\infty} e^{-\pi n^2 e^{4\lambda}}\right\} d\lambda,$$

by integration by parts. And as in (1) the subject of integration is an even function of λ.

$$2x^{2n}\psi(x, y) = -x^{2n}\int_{-\infty}^{\infty} \sin x\lambda . \chi(\lambda)\,d\lambda,$$

$$= (-)^{n+1}\int_{-\infty}^{\infty} \sin x\lambda . \chi^{(2n)}(\lambda)\,d\lambda,$$

Hence
$$0 = \int_{-\infty}^{\infty} \cos x\lambda . \chi^{(2n)}(\lambda)\,d\lambda.$$

$$2x^{2n}\phi(x, y)\cos(\beta - \iota\alpha)\,x = (-)^n\int_{-\infty}^{\infty} \cos(x - \beta + \iota\alpha)\,\lambda . \theta^{(2n)}(\lambda)\,d\lambda$$

$$= (-)^n\int_{-\infty}^{\infty} \cos(x - \beta)\,\lambda . \theta^{(2n)}(\lambda - \iota\alpha)\,d\lambda,$$

by an evident contour integration, provided that

$$-\frac{\pi}{8} < \alpha < \frac{\pi}{8} \quad\ldots\ldots\ldots\ldots\ldots\ldots\ldots (4).$$

Similarly, under the same restriction as regards α,

$$2x^{2n}\psi(x, y)\sin(\beta - \iota\alpha)\,x = (-)^{n-1}\int_{-\infty}^{\infty} \cos(x - \beta)\lambda . \chi^{(2n)}(\lambda - \iota\alpha)\,d\lambda.$$

And we have immediately, by Fourier's theorem,

$$\left.\begin{aligned} \frac{2}{\pi}\int_0^{\infty} x^{2n}\phi(x, y)\cos(\beta - \iota\alpha)\,x\,dx &= (-)^n\,\theta^{(2n)}(\beta - \iota\alpha) \\ \frac{2}{\pi}\int_0^{\infty} x^{2n}\psi(x, y)\sin(\beta - \iota\alpha)\,x\,dx &= (-)^{n+1}\chi^{(2n)}(\beta - \iota\alpha) \end{aligned}\right\} \ldots (5).$$

Mr. Hardy's equation (3), p. 1013, is obtained by putting $y = 0$, $\beta = 0$.

It is easy to verify that, on account of (2), $\theta(\lambda)$ steadily decreases as λ increases from 0 to ∞. And it readily follows that $(-)^n\theta^{(2n)}(0)$ is positive, while $\theta^{(2n)}(\beta)$ vanishes for n values of β between 0 and ∞. Further the radius of convergence of the power series for $\theta(\lambda)$ is clearly $\pi/8$; hence on account of (4) we may expand $(-)^n\theta^{(2n)}(\iota\alpha)$ in powers of α and every term will be positive, and therefore

$$\int_0^{\infty} x^{2n}\phi(x, y)\cosh\alpha x\,dx$$

is essentially positive so long as the inequality (4) is satisfied.

In the particular case when $\beta = 0$ and $n = 0$, equations (5) become

$$\frac{2}{\pi}\int_0^{\infty} \phi(x, y)\cosh\alpha x\,dx = f(\iota\alpha)\cos y\alpha$$

$$= 2\cos\alpha\cos y\alpha - e^{\iota\alpha}\cos y\alpha\left\{1 + 2\sum_{n=1}^{\infty} e^{-\pi n^2(\cos 4\alpha + \iota\sin 4\alpha)}\right\},$$

$$\frac{2}{\pi}\int_0^\infty \psi(x, y)\sinh \alpha x\, dx = -f(\iota\alpha)\sin y\alpha$$

$$= -2\cos\alpha\sin y\alpha + e^{\iota\alpha}\sin y\alpha\left\{1 + 2\sum_{n=1}^{\infty} e^{-\pi n^2(\cos 4\alpha + \iota\sin 4\alpha)}\right\}.$$

Since

$$(-)^n \frac{d^{2n}}{d\alpha^{2n}}(2\cos\alpha\cos y\alpha) = (1+y)^{2n}\cos(1+y)\alpha + (1-y)^{2n}\cos(1-y)\alpha,$$

and

$$(-)^n \frac{d^{2n}}{d\alpha^{2n}}(2\cos\alpha\sin y\alpha) = (1+y)^{2n}\sin(1+y)\alpha - (1-y)^{2n}\sin(1-y)\alpha,$$

are both of constant sign we see, on making $\alpha \to \pi/8$, and following Mr. Hardy's argument precisely, that for any value of y in the strip (2) of the z-plane both

$$\phi(x, y) = 0 \text{ and } \psi(x, y) = 0$$

have an infinite number of real roots when regarded as equations to determine x.

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DETERMINANTS OF CYCLICALLY REPEATED ARRAYS.

By *Prof. W. Burnside.*

In a recent paper* Sir Thomas Muir has shown that when the number, n, of arrays is equal to the number of lines (or columns) in each array, the determinant can be expressed as the product of n determinants, in whose elements the original elements enter linearly. The result is, in fact, true without limitation; and may be proved by an obvious extension of the method which exhibits a circulant as the product of its linear factors.

Let D denote the circulant of the n m-line arrays

$$\begin{array}{llll} a^r_{11}, & a^r_{12}, & \ldots, & a^r_{1m}, \\ a^r_{21}, & a^r_{22}, & \ldots, & a^r_{2m}, \quad r = 1, 2, \ldots, n. \\ a^r_{m1}, & a^r_{m2}, & \ldots, & a^r_{mm}. \end{array}$$

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* *Messenger of Mathematics*, vol. xlv., p. 142.

Take w an assigned primitive n^{th} root of unity, and put

$$\sum_{t=1}^{t=n} w^{-(s-1)(t-1)} a^t_{ij} = A^s_{ij},$$

so that
$$na^t_{ij} = \sum_{s=1}^{s=n} w^{(s-1)(t-1)} A^s_{ij}.$$

The elements of the $[(p-1)m+q]^{\text{th}}$ row of D are

$$a^p_{q1},\ a^p_{q2},\ \ldots,\ a^p_{qm},\ a^{p+1}_{q1},\ \ldots,\ a^{p+1}_{qm},\ a^{p+2}_{q1},\ \ldots,\ a^{p-1}_{qm},$$

where the a_{ij}'s are to be replaced by the A_{ij}'s by the preceding formula.

The terms containing the upper suffix s that occur are

$$\frac{1}{n} w^{(s-1)(t-1)} \times$$

$$(A^s_{q1},\ A^s_{q2},\ \ldots,\ A^s_{qm},\ w^s A^s_{q1},\ w^s A^s_{q2},\ \ldots,\ w^s A^s_{qm},\ w^{2s} A^s_{q1},\ \ldots,\ w^{-s} A^s_{qm}).$$

The terms that contain the upper suffix s occurring in the $[(p'-1)m+q]^{\text{th}}$ row differ from the above only by the outside factor. Hence it follows that D can be expressed as the sum of a number of determinants of nm rows and columns each of which is a numerical multiple of the determinant whose $[(x-1)m+y]^{\text{th}}$ row is

$$A^x_{y1},\ A^x_{y2},\ \ldots,\ A^x_{ym},\ w^x A^x_{y1},\ \ldots,\ w^x A^x_{ym},\ w^{2x} A^x_{y1},\ \ldots,\ w^{-x} A^x_{ym};$$

so that D itself is a numerical multiple of this determinant.

Replace the $[(n-1)m+i]^{\text{th}}$ column of this determinant by

$$i^{\text{th}} \text{ column} + w^{-x}(m+i)^{\text{th}} \text{ column} + w^{-2x}(2m+i)^{\text{th}} \text{ column}$$
$$+ \ldots + w^x[(n-1)m+i]^{\text{th}} \text{ column}$$

for each i from 1 to m; thereby affecting the determinant only by a numerical factor.

The elements of the last m columns will then all be zeros, except those that belong to the rows from the $[(x-1)m+1]^{\text{th}}$ to the xm^{th}; and these will be

$$\begin{matrix} nA^x_{11}, & nA^x_{12}, & \ldots, & nA^x_{1m}, \\ nA^x_{21}, & nA^x_{22}, & \ldots, & nA^x_{2m}, \\ \cdots & \cdots & \cdots & \cdots \\ nA^x_{m1}, & nA^x_{m2}, & \ldots, & nA^x_{mm}. \end{matrix}$$

Hence, apart from numerical factors, D is divisible by

$$\begin{vmatrix} A^x_{11}, & A^x_{12}, & \ldots, & A^x_{1m} \\ A^x_{21}, & A^x_{22}, & \ldots, & A^x_{2m} \\ \cdots & \cdots & \cdots & \cdots \\ A^x_{m1}, & A^x_{m2}, & \ldots, & A^x_{mm} \end{vmatrix}$$

for each x.

No two of these determinants can have common factors for arbitrary values of the a_{ij}'s, since when each of the n m-line arrays is itself a circulant the nm linear factors of the determinants are known to be all distinct. Hence

$$D = N . \prod_{x=1}^{x=n} \begin{vmatrix} A^x_{11}, & A^x_{12}, & \ldots, & A^x_{1m} \\ A^x_{21}, & A^x_{22}, & \ldots, & A^x_{2m} \\ \cdots & \cdots & \cdots & \cdots \\ A^x_{m1}, & A^x_{m2}, & \ldots, & A^x_{mm} \end{vmatrix}$$

where N is numerical. It is easy to show by comparing coefficients that $N = \pm 1$ according to the order in which the rows of D are written.

FACTORISATION OF $N=(x^y \mp y^x)$.

By *Lt.-Col. Allan Cunningham, R.E.*, Fellow of King's College, London.

[The Author is indebted to Mr. H. J. Woodall for help in reading the proof sheets.]

1. *Introduction.* THE numbers, whose factorisation is considered in this Paper, are of form

$$N = x^y - y^x, \quad N' = x^y + y^x \ldots\ldots\ldots (1).$$

It will be supposed throughout that

$$x \text{ and } y > 1\,; \quad \text{and } x \text{ prime to } y \ldots\ldots\ldots (1a).$$

These numbers rise so rapidly as x, y increase that complete factorisation (into prime factors) is possible only for a very small range of x, y; in fact—

$$2^{25} \mp 25^2,\ 4^{13} \mp 13^4,\ 8^9 \mp 9^8,\ 16^3 \mp 3^{16},\ 5^{11} \mp 11^5,\ 10^7 \mp 7^{10},\ \&c.$$

are beyond the powers of the present large Factor-Tables.

2. *Algebraic Factors.* When x, y have certain forms, then N or N' is algebraically resolvable into two or more factors. These cases are—

i. Difference of Squares. ii. Binomial Factors. iii. Aurifeuillians.

3. *Difference of Squares.* N is algebraically resolvable at sight into a continued product of $(\alpha+1)$ factors, when one of x, y is of form $(2h)^e$, where $e=2^\alpha$.

Ex. $x=2^2$ gives $N=4^y-y^4=(2^y-y^2)\,||\,(2^y+y^2)$......................$(2a)$,

$x=2^4$ gives $N=16^y-y^{16}=(2^y-y^4)\,|\,(2^y+y^4)\,||\,(2^{2y}+y^8)$.......$(2b)$.

These are the only cases worth recording: as, when either $h>1$, or $\alpha>2$, the numbers N are too high* to admit of complete factorisation. A number of examples of these two cases completely factorised are given in the Table at the end of this Paper.

The first case $N=(4^y \sim y^4)$ has the peculiarity that its two algebraic co-factors $(2^y \sim y^2)$, (2^y+y^2) are themselves of the form N, N' of (1). This is the only case possessing this property.

4. *Binomial Factors.* Each of N, N' contains an obvious algebraic binomial factor when one of x, y is of form $(nh)^n$ with n odd and >1.

Ex. Take $x=2^a$, $y=3^3$, $[n=3,\ h=1]$; then

$$N \text{ or } N'=(2^a)^{27}\mp 27^{2^a}=(2^{9a})^3\mp(3^{2^a})^3,$$

which contain the obvious factor $(2^{9a}\mp 3^{2^a})$..................................(3).

This is the only form worth record: as when either $h>1$, or $n>3$, the numbers N, N' are too high† to admit of complete factorisation. Examples of these forms with $x=2$, 4, 16, $y=27$ will be found in the Table at the end.

5. *Aurifeuillians.* These may be of three different orders (n), which must be separately considered.

i. $n=2$. ii. $n=\omega$ (odd). iii. $n=2\omega$ (twice an odd number)...(4).

6. *Bin-Aurifeuillians.* These are numbers of form

$$N'=4X^4+Y^4 \;\ldots\ldots\ldots\ldots\ldots\ldots(5).$$

which are algebraically resolvable into two co-factors (say L, M), viz.

$$N'=L.M=(Y^2-2XY+2X^2)(Y^2+2XY+2X^2) \;\ldots\ldots\ldots(5a).$$

* The smallest of these is $N=(36^5 \sim 5^{36})$.
† The smallest of these is $(5^{27}\mp 27^5)$.

The conditions for x, y that $N' = x^y + y^x$ may be expressible in the above form (5) are

$$x = 4h^4,\quad y = 2\eta + 1 \text{ (an \textit{odd} number)} \ldots\ldots\ldots\ldots (6).$$

whereby

$$\begin{aligned} N' &= (4h^4)^y + y^{4h^4} \\ &= 4(2^\eta h^y)^4 + (y^{h^4})^4;\quad [X = 2^\eta h^y,\ Y = y^{h^4}] \ldots\ldots\ldots\ldots (6a). \end{aligned}$$

Ex. $\quad x = 4,\ y = 2\eta + 1$ (odd), $h = 1$.

$$\begin{aligned} N' &= 4^y + y^4 = 4.2^{4\eta} + y^4 \\ &= L.M = (y^2 - 2^{\eta+1}y + 2^{2\eta+1}).(y^2 + 2^{\eta+1}y + 2^{2\eta+1}) \ldots\ldots\ldots (6b). \end{aligned}$$

This is the only Bin-Aurifeuillian form of N' worth detailing: as when $h > 1$, the numbers N' are too high* for complete factorisation. A number of examples (with $x = 4$, $y = 3$ to 27) completely factorised are given in the Table at the end.

6a. *Bin-Aurifeuillians as factors of N.* When

$$x = 2^4h^8,\quad y = 2\eta + 1 \text{ (odd)}$$

then

$$\begin{aligned} N &= x^y - y^x = (2^4h^8)^y - y^{16h^8} \\ &= (2^y h^{2y} - y^{4h^8}) \,\|\, (2^y h^{2y} + y^{4h^8}) \,\|\, (2^{2y} h^{4y} + y^{8h^8}) \ldots\ldots\ldots\ldots (7). \end{aligned}$$

Here the largest factor (say Z) is

$$Z = 2^{2y} h^{4y} + y^{8h^8} = 2^{4\eta+2} h^{4y} + y^{8h^8} = 4X^4 + Y^4 \ldots\ldots\ldots\ldots (7a),$$

which is a Bin-Aurifeuillian (see 5), and therefore resolvable as in (5a).

Ex. $\quad x = 16,\ y = 2\eta + 1$ (odd), $h = 1$.

then

$$\begin{aligned} Z &= 2^{2y} + y^8 = 4.2^{4\eta} + (y^2)^4 \\ &= L.M = (y^4 - 2^{\eta+1}y^2 + 2^{2\eta+1})\,(y^4 + 2^{\eta+1}y^2 + 2^{2\eta+1}) \ldots\ldots\ldots (7b). \end{aligned}$$

This is the only case of N possessing a Bin-Aurifeuillian as an algebraic factor, which is worth detailing here: as when $h > 1$, the numbers Z are too high† for complete factorisation. A number of examples (with $x = 16$, $y = 3$ to 27) completely factorised are given in the Table at the end.

7. *Aurifeuillians of odd order* (n). These are numbers of form

$$N = (X^n - Y^n) \div (X - Y),\ \text{with } n = 4i + 1 \ldots\ldots\ldots\ldots (8a),$$

$$N' = (X^n + Y^n) \div (X + Y),\ \text{with } n = 4i + 3 \ldots\ldots\ldots\ldots (8b),$$

along with the condition $\quad nXY = \square \ldots\ldots\ldots\ldots (9).$

* The smallest of these is $N' = (64^3 + 3^{64})$ given by $h = 2$.
† The smallest of these is $Z = 2^{18} + 3^{2048}$.

Each of these is algebraically expressible as a difference of squares, and therefore resolvable into two co-factors (say L, M). The simplest forms of X, Y satisfying the condition (9) are

$$X = H^2, \quad Y = nK^2 \quad\ldots\ldots(9a).$$

By taking x, y of forms

$$x = (2h+1)^2,\ odd; \quad y = n^n k^{2n},\ [n \text{ odd}] \quad\ldots\ldots(9b).$$

the numbers N, $N' = x^y \mp y^x$ can be expressed in the forms $X^n \mp Y^n$ along with the condition $nXY = \square$, and will then be divisible by $(X \mp Y)$. The co-factors will be the Aurifeuillians **N**, **N'**.

It is not worth while developing this further, as the smallest numbers of this kind are too high for complete factorisation. The smallest example of each kind is shown below.

1°. $N' = 25^{27} + 27^{25} = 5^{54} + 3^{75}$, which contains $(5^{18} + 3^{25})$,

and the co-factor **N'** is seen to be a Trin-Aurifeuillian

$$N' = \frac{5^{54} + 3^{75}}{5^{18} + 3^{25}} = (5^{18} - 3.3^{12}.5^9 + 3^{25})(5^{18} + 3.3^{12}.5^9 + 3^{25}).$$

2°. $N = 9^{3125} - 3125^9 = 3^{6250} - 5^{45}$, which contains $(3^{1250} - 5^9)$,

and the co-factor **N** is seen to be a *Quint-Aurifeuillian*,

$$\mathbf{N} = \frac{3^{6250} - 5^{45}}{3^{1250} - 5^9} = (3^{2500} + 3.3^{1250}.5^9 + 5^{18})^2 - (5^5.3^{625})^2.(3^{1250} + 5^9)^2.$$

8. *Aurifeuillians of even order* $(n = 2\omega)$. These are numbers of form

$$\mathbf{N}' = (X^{2n'} + Y^{2n'}) \div (x^2 + y^2), \text{ with } n' \text{ odd} \quad\ldots\ldots(10),$$

and with the condition $\quad 2n'XY = \square \quad\ldots\ldots(11).$

These are algebraically expressible as a difference of squares, and are therefore resolvable into two co-factors (say L, M). The simplest forms of X, Y satisfying the condition (11) are

$$X = H^2, \quad Y = 2n'K^n \quad\ldots\ldots(11a).$$

By taking x, y of forms

$$x = (2h+1)^2\ odd; \quad y = (2n')^{2n'}.k^{4n'}\ [n'\ odd] \quad\ldots\ldots(11b),$$

the numbers $N' = x^y + y^x$ can be expressed in the form $(X^{2n'} + Y^{2n'})$ along with the condition $2n'XY = \square$, and will then be divisible by $(X^2 + Y^2)$. The co-factor will be the Aurifeuillian **N'**.

It is not worth while developing this further as the smallest numbers of this kind are too high for complete factorisation.

The smallest is

$$N' = 25^{46656} + 46656^{25} = 5^{6.7776} + 6^{6.25},$$

which contains $(5^{2.7776} + 6^{2.25})$ and the co-factor $\mathbf{N}'$ is seen to be a *Sext-Aurifeuillian.*

$$\mathbf{N}' = \frac{5^{6.7776} + 6^{6.25}}{5^{2.7776} + 6^{2.25}} = (5^{2.7776} + 3.5^{7776}.6^{25} + 6^{2.25})^2 - 6.6^{25}.5^{7776}(5^{7776} + 6^{25})^2.$$

9. *Use of Numerical Canons.* The factorisation of large numbers $N, N' > 10^7$, wherein the elements x, y are powers of 2, 3, 5, 7, 10, 11, has been rendered possible by the help of certain Numerical Canons (Binary, Ternary, &c.) which have been compiled* by the author.

These give the Residues, both + and − of the powers (n), of the above bases (2, 3, &c.) up to the limits named below,

Residues of	2^n;	$3^n, 5^n, 7^n, 10^n, 11^n$,
Limit of n	100;	30,

after division by every prime (p) and prime-power $p^k \ngtr 10000$.

10. *Perfect Squares and Powers.* No perfect squares or powers have as yet been found among these numbers N, N': so that it would seem probable that none exist.

11. *Dimorphism.* No case is known of any number being expressible in two ways in the same form N or N': and only one case is as yet known of a number being expressible in both forms N, N', viz.

$$17 = 3^4 - 4^3 = 3^2 + 2^3.$$

If, however, the value $y=1$ be admitted—hitherto excluded, see (1*a*)—every number N or N' would be expressible in two ways in the same form, and every number whatever (say Z) would be expressible in both forms, for

$$N = (N+1)^1 - 1^{N+1} \text{ and } N' = (N'-1)^1 + 1^{N'-1};$$

$$Z = (Z+1)^1 - 1^{Z+1} = (Z-1)^1 + 1^{Z-1}.$$

12. *Factorisation Tables.* Here follow two Tables giving the factorisation of the numbers N, N' in separate Tables.

1°. *Arrangement of Factors.* Each number N, N' is shown resolved first into its Algebraic Prime Factors (A.P.F.) and *Aurifeuillian Factors* (L, M): these are arranged in order of magnitude, the smallest on the left, the highest on the right, and are separated by special symbols.

Each A.P.F., and each L, M is shown resolved as far as possible into its numerical prime and prime-power factors (p and p^n): these are arranged in order of magnitude of the primes, the lowest on the left, the highest on the right.

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* At present only in MS. Those for bases 2, 10 were compiled by Mr. H. J. Woodall and the author jointly (not independently). The rest are due to the author.

The powers of the small primes $\ngtr 11$ are printed in a condensed form, thus :—

4, 8, 16, 32, &c.; 9, 27, 81, 729, &c,; 5, 25, 125, &c.; 49, &c.; 121, &c.

2°. *Special multiplication-symbols* (. | ‖ ; :). These are used to separate various kinds of factors in such a way as to indicate the nature of the factors.

Use of dot (.). This is used between arithmetical factors in the same A.P.F., L, or M (but not between the A.P.F. or L, M themselves).

A dot on the right of an arithmetical factor, followed by a blank, indicates the existence of an other arithmetical factor of unknown constitution.

Use of bars (| and ‖). These are used between the A.P.F. of (X^2-Y^2), (X^4-Y^4), &c.; thus

$$X^2-Y^2=(X-Y)\,\|\,(X+Y)\,; \quad X^4-Y^4=(X-Y)\,|\,(X+Y)\,\|\,(X^2+Y^2)\,;$$

the double bar (‖) being placed just before the highest A.P.F.

Use of semi-colon (;). This is used between the A.P.F. of $(X^n \mp Y^n)$ where n is *odd*, thus

$$X^n \mp Y^n=(X \mp Y)\,;\ (X^{n-1} \pm X^{n-2}Y+\ \&c.).$$

A semi-colon on the extreme right indicates the complete factorisation of the highest A.P.F.

Use of colon (:). This is used between the twin "Aurifeuillian Factors" (L, M) of an Aurifeuillian. These Aurifeuillians occur as complete A.P.F., so that their ends are marked by either bars (|) or semi-colons (;)—[see above].

3°. *Symbols* († ‡). These symbols are used (in incomplete factorisations) to show the limit to which the search for factors has been carried, thus

† to 1000, ‡ to 10000 [or a little further].

4°. *Use of queries* (?). A query (?) on the right of a large arithmetical factor ($>10^7$) indicates that this factor is beyond the power of the Tables to resolve.

13. *Table of* $x^{x+1} \mp (x+1)^x$. The case of $y-x=1$ seems of some special interest. Accordingly the short Table below (extracted from the larger general Tables) gives the factorisation of

$$N=x^{x+1}-(x+1)^x, \quad N'=x^{x+1}+(x+1)^x.$$

<table>
<tr><th>x, y</th><th>N = x^y − y^x.</th><th>N' = x^y + y^x.</th></tr>
<tr><td>1, 2</td><td>−1;</td><td>3;</td></tr>
<tr><td>2, 3</td><td>−1;</td><td>17;</td></tr>
<tr><td>3, 4</td><td>1 ‖ 17;</td><td>5 : 29;</td></tr>
<tr><td>4, 5</td><td>79 ‖ 3.59;</td><td>17 : 97;</td></tr>
<tr><td>5, 6</td><td>47.167;</td><td>7.3343;</td></tr>
<tr><td>6, 7</td><td>162287;</td><td>5.131.607;</td></tr>
<tr><td>7, 8</td><td>23.159463;</td><td>3.11.19.12539;</td></tr>
<tr><td>8, 9</td><td>257.354751;</td><td>‡</td></tr>
<tr><td>9, 10</td><td>‡</td><td>11. ‡</td></tr>
<tr><td>10, 11</td><td>3.37.53.12589253;</td><td>2531.49757971;</td></tr>
<tr><td>15, 16</td><td>7.2551 | 89.937 ‖ 25793 : 277.509;</td><td></td></tr>
</table>

Table A.

<table>
<tr><th>x, y</th><th>N.</th></tr>
<tr><td>2, 3</td><td>−1;</td></tr>
<tr><td>5</td><td>7;</td></tr>
<tr><td>7</td><td>79;</td></tr>
<tr><td>9</td><td>431;</td></tr>
<tr><td>11</td><td>41.47;</td></tr>
<tr><td>13</td><td>71.113;</td></tr>
<tr><td>15</td><td>7.4649;</td></tr>
<tr><td>17</td><td>130783;</td></tr>
<tr><td>19</td><td>523927;</td></tr>
<tr><td>21</td><td>641.3271;</td></tr>
<tr><td>23</td><td>7.1198297;</td></tr>
<tr><td>25</td><td>7.167.28703;</td></tr>
<tr><td>27</td><td>503; 7.38119;</td></tr>
<tr><td>29</td><td>23.97.240641;</td></tr>
<tr><td>2, 31</td><td>7.17.47.599.641;</td></tr>
<tr><td>3, 5</td><td>2.59;</td></tr>
<tr><td>7</td><td>4.461;</td></tr>
<tr><td>11</td><td>8.21977;</td></tr>
<tr><td>3, 13</td><td>2.796063;</td></tr>
<tr><td>4, 3</td><td>−1||17;</td></tr>
<tr><td>5</td><td>7||3.19;</td></tr>
<tr><td>7</td><td>79||3.59;</td></tr>
<tr><td>9</td><td>431||593;</td></tr>
<tr><td>11</td><td>41.47||9.241;</td></tr>
<tr><td>13</td><td>71.113||9.929;</td></tr>
<tr><td>15</td><td>7.4649||32993;</td></tr>
<tr><td>17</td><td>130783||3.43787;</td></tr>
<tr><td>19</td><td>523927||3.179.977;</td></tr>
<tr><td>21</td><td>641.3271||2097593;</td></tr>
<tr><td>23</td><td>7.1198297||3.67.41737;</td></tr>
<tr><td>25</td><td>7.167.28703||3.11185019;</td></tr>
<tr><td>27</td><td>503;7.38119||521;73.3529;</td></tr>
<tr><td>29</td><td>23.97.240641||27.11.41.44089;</td></tr>
<tr><td>4, 31</td><td>7.17.47.599.641||27.79536467?‡</td></tr>
<tr><td>5, 7</td><td>2.23.31.43;</td></tr>
<tr><td>9</td><td>4.473519;</td></tr>
<tr><td>5, 11</td><td>2.3.23.36373;</td></tr>
<tr><td>6, 5</td><td>−47.167;</td></tr>
<tr><td>7</td><td>162287;</td></tr>
<tr><td>6, 11</td><td>5. †</td></tr>
<tr><td>7, 9</td><td>2.103.172673;</td></tr>
<tr><td>7, 11</td><td>4.13.19.1981619;</td></tr>
<tr><td>8, 3</td><td>−23.263;</td></tr>
<tr><td>5</td><td>−23.15559;</td></tr>
<tr><td>7</td><td>−23.159463;</td></tr>
<tr><td>9</td><td>257.354751;</td></tr>
<tr><td>8, 11</td><td>‡</td></tr>
<tr><td>9, 11</td><td>‡</td></tr>
<tr><td>10, 3</td><td>−58049;</td></tr>
<tr><td>7</td><td>−3.90825083;</td></tr>
<tr><td>9</td><td>‡</td></tr>
<tr><td>10, 11</td><td>3.37.53.12589253;</td></tr>
<tr><td>12, 5</td><td>23.</td></tr>
<tr><td>14, 3</td><td>25.107.1787;</td></tr>
<tr><td>16, 3</td><td>−73|89||53:125;</td></tr>
<tr><td>5</td><td>−593|9.73||457:857;</td></tr>
<tr><td>7</td><td>−2273|9.281||5.349:3313;</td></tr>
<tr><td>9</td><td>−23.263|11.643||6481:5.1933;</td></tr>
<tr><td>11</td><td>−49.257|3.5563||
||5.1789:53.461;</td></tr>
<tr><td>13</td><td>−20369|3.1251||
||15121:11677;</td></tr>
<tr><td>15</td><td>−7.2551|89.937||
||25793:277.509;</td></tr>
<tr><td>17</td><td>7.6793|3.233 307||
||125.533:362561;</td></tr>
<tr><td>19</td><td>7.23.2447|3.113.1931||
||5.56989:109.9397;</td></tr>
<tr><td>21</td><td>1902671|1097.2089||
||5.13.41.521:3194801;</td></tr>
<tr><td>23</td><td>73.113.983|9.227.4243||
||5.1300333:10835233?†</td></tr>
<tr><td>25</td><td>3316807|9.3771673||
||53.113.4813:39065057?‡</td></tr>
<tr><td>16, 27</td><td>431;7.73.607|593;2272233||
||881;139393:5.61;13.36997;</td></tr>
</table>

Table B.

x, y	N′.
2, 3	17;
5	3.19;
7	3.59;
9	593;
11	9.241;
13	9.929;
15	32993;
17	3.43787;
19	3.179.977;
21	2097593;
23	3.67.41737;
25	3.11185019;
27	521;73.3529;
29	27.11.41.44089;
2, 31	27.79536467?‡
3, 5	16.23;
7	2.5.11.23;
11	2.233.383;
3, 13	8.5.167.239;
4, 3	5:29;
5	17:97;
7	5.13:17.17;
9	5.61:881;
11	5.293:13.13.17;
13	37.181:25.401;
15	29153:36833;
17	173.709:5.109.257;
19	13.38861:5.108821;
21	5.13.73.433:2140601;
23	17.409.1193:5.13.130513;
25	33350257?‡:29.373.3121;
4, 27	25.17:617;157.1381:13.24373;
5, 7	4.81.293;
9	2.1006087;
5, 11	8.31.251.787;
6, 5	7.3343;
7	5.131.607;
6, 11	7. †

x, y	N′.
7, 9	64.705259;
7, 11	2.3.17.19576607;
8, 3	11.643;
5	3.141131;
7	3.11.19 12539;
9	‡
11	3.19.43.1193 3011;
9, 11	4.25.367.919319;
10, 3	11.53.103;
7	11.4397.6047;
9	11. ‡
10, 11	2531.49757971;
12, 5	13.19.463 2137;
14, 3	677.7069;
16, 3	17.2532401;
5	17. ‡
16, 7	17. ‡

END OF VOL. XLV.

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